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If a system has at least one solution, then it is said to be *consistent*. If a system has no solutions, then it is *inconsistent*.

A *consistent* system has either

1. exactly one solution (no free variables),
2. infinitely many solutions (because there is at least one free variable).

A matrix is in *reduced row echelon form* if all of the following conditions are satisfied:

1. If a row has nonzero entries, then the first nonzero entry (i.e., *pivot*) is 1.
2. If a column contains a pivot, then all other entries in that column are zero.
3. If a row contains a pivot, then each row above contains a pivot further to the left.

A set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ is *linearly independent* if the only solution to the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_p = \mathbf{0}$$

is the trivial solution (all constants equal zero). If there is a nontrivial solution, the vectors are *linearly dependent*.

A *linear combination* is a weighted sum of vectors (where the weights are scalars). For example, if $c_1, \ldots, c_p$ are constants, then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

is a *linear combination* of $\mathbf{v}_1, \ldots, \mathbf{v}_p$.

A function T that maps vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called a *linear transformation* if for all vectors $\mathbf{x}$, $\mathbf{y}$ and scalars c,

1. $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$,
2. $T(c\mathbf{x}) = cT(\mathbf{x})$.

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has an m-by-n matrix A called the *standard matrix* of T such that $T(\mathbf{x}) = A\mathbf{x}$.

The *span* of a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is the set of all possible linear combinations of those vectors.

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is *onto* if every $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

If T has standard matrix A, then the following are equivalent.

1. T is onto,
2. The columns of A span $\mathbb{R}^m$,
3. A has a pivot in every row.

The *standard matrix* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is an m-by-n matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

The i^{th} column A is $T(\mathbf{e}_i)$ where $\mathbf{e}_i$ is the i^{th} elementary basis vector for $\mathbb{R}^n$.

A *subspace* is a subset of a vector space that is

1. Closed under addition, and
2. Closed under scalar multiplication.

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is *one-to-one* if every $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

If T has standard matrix A, then the following are equivalent.

1. T is one-to-one,
2. The columns of A are linearly independent,
3. A has a pivot in every column.

DEFINITION *Basis* LINEAR ALGEBRA	DEFINITION & CONCEPT *Dimension* LINEAR ALGEBRA
DEFINITION & CONCEPT *Rank* LINEAR ALGEBRA	DEFINITION & CONCEPT *Column Space* LINEAR ALGEBRA
DEFINITION *Null Space* LINEAR ALGEBRA	THEOREM *Rank + Nullity Theorem* LINEAR ALGEBRA
DEFINITION *Eigenvectors and Eigenvalues* LINEAR ALGEBRA	DEFINITION *Eigenspace* LINEAR ALGEBRA
DEFINITION *Characteristic Polynomial* LINEAR ALGEBRA	DEFINITION & CONCEPT *Diagonalizable* LINEAR ALGEBRA

The *dimension* of a vector space V is the number of vectors in a basis for V.

It is a theorem that all bases for a vector space have the same number of elements.

A *basis* of a vector space V is set that is

1. Linearly independent, and
2. Spans V.

The *column space* of a matrix is the span of the columns of A.

The column space is also the range of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.

The *rank* of a matrix is the number of pivots in the reduced row echelon form of the matrix.

The rank of a matrix is also

1. The dimension of the column space.
2. The dimension of the row space.

For any m-by-n matrix A, the rank of A plus the nullity of A (number of pivots plus the number of free variables) is always n.

The *null space* of a matrix is the set of all vectors $\mathbf{x}$ such that $A\mathbf{x} = 0$.

For a matrix A with eigenvalue λ, the *eigenspace* of A corresponding to λ is the set of all eigenvectors of A corresponding to λ.

For an n-by-n matrix A, a nonzero vector $\mathbf{x}$ and a scalar λ are eigenvectors and eigenvalues (resp.) of A if $A\mathbf{x} = \lambda\mathbf{x}$.

A matrix A is *diagonalizable* if there is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

The columns of P are eigenvectors of A and the diagonal entries of D are the eigenvalues of A.

For any square matrix A, the *characteristic polynomial* is $\det(A - \lambda I)$.

The roots of the characteristic polynomial are the eigenvalues of A.

Definition *Inner Product* Linear Algebra	Definition *Norm* Linear Algebra
Definition *Orthogonal and Orthonormal Sets* Linear Algebra	Definition *Orthogonal Complement* Linear Algebra
Theorem *Fundamental Theorem of Linear Algebra* Linear Algebra	Theorem *Orthogonal Decomposition Theorem* Linear Algebra
Definition *Normal Equations* Linear Algebra	Definition & Concept *Orthogonal Matrix* Linear Algebra
Definition *Orthogonally Diagonalizable* Linear Algebra	Definition & Concept *Symmetric Matrix* Linear Algebra

The *norm* of a vector $\mathbf{x}$ is $||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

Norm is also known as length.

The *inner product* of two vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$ is $\mathbf{x}^T\mathbf{y}$. It is also known as the dot-product ($\mathbf{x} \cdot \mathbf{y}$).

For any set of vectors W in $\mathbb{R}^n$, the *orthogonal complement* of W (denoted $W^\perp$) is the set of all vectors in $\mathbb{R}^n$ orthogonal to everything in W.

An orthogonal complement is always a subspace of $\mathbb{R}^n$.

A set of vectors is *orthogonal* if every vector in the set is orthogonal to every other vector in the set.

A set of vectors is *orthonormal* if it is orthogonal and every vector in the set also has norm 1.

For any subspace W in $\mathbb{R}^n$ and any vector $\mathbf{y}$ in $\mathbb{R}^n$, there is a unique decomposition $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^\perp$.

We call the vector $\hat{\mathbf{y}}$ the *orthogonal projection* of $\mathbf{y}$ onto W. It is the vector in W that is closest to $\mathbf{y}$.

For any m-by-n matrix A,

1. $(\text{Row}\, A)^\perp = \text{Nul}\, A$,
2. $(\text{Col}\, A)^\perp = \text{Nul}\, A^T$.

An n-by-n matrix Q is *orthogonal* if the columns of Q are an orthonormal set.

The inverse of an orthogonal matrix is its transpose: $Q^{-1} = Q^T$.

For an inconsistent linear system $A\mathbf{x} = \mathbf{b}$, you can find a least-squares solution by solving the *normal equations* for the system:

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

Note: If the columns of A are linearly independent, then $A^T A$ is invertible.

A matrix A is *symmetric* if $A^T = A$.

Symmetric matrices have the following properties.

1. All eigenvalues are real.
2. The matrix can be orthogonally diagonalized.

A matrix A is *orthogonally diagonalizable* if there is an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$.

Matrices with real eigenvalues are orthogonally diagonalizable if and only if they are symmetric.