Geometric Series

- **Standard Form** $\sum_{n=0}^{\infty} ar^n$ where a is the first term, and r is the common ratio.
- **Test for Convergence** A geometric series converges if $|r| < 1$, otherwise it diverges.
- **Formula for the Sum** $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if the geometric series converges.
- **Example** Zeno's series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1$. Converges because the common ratio $r = \frac{1}{2}$.

p-Series

- **Standard Form** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p is a constant.
- **Test for Convergence (p-Test)** A p-series converges if $p > 1$, otherwise it diverges.
- **Example 1** Harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$. Diverges by the p-test because $p = 1$.
- **Example 2** Basel series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$. Converges by the p-test because $p = 2$.

Alternating Series

- **Standard Form** $\sum_{n=0}^{\infty} (-1)^n b_n$ where each $b_n > 0$.
- **Test for Convergence (Alternating Series Test)** An alternating series converges if
  1. $b_{n+1} \leq b_n$ for all n (decreasing), and
  2. $\lim_{n\to\infty} b_n$ (terms approach zero).
- **Error Formula** $|S_n - S_\infty| \leq b_{n+1}$ where S_n is the n^{th} partial sum and S_∞ is the infinite sum.
- **Example 1** Grandi's series $1-1+1-1+1-1+\ldots$. Diverges because the partial sums don't converge.
- **Example 2** Alternating Harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$. Converges by the Alternating Series Test.

Comparison Test

- For two series $\sum a_n$ and $\sum b_n$ with positive terms, if $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is a positive number, then $\sum a_n$ and $\sum b_n$ are *comparable.* This means they either both converge or both diverge.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ or if $b_n > a_n$ for all n, then $\sum b_n$ *dominates* $\sum a_n$, which means two things:
  1. If $\sum b_n$ converges, then so does the smaller sum $\sum a_n$.
  2. If $\sum a_n$ diverges, then so does the bigger sum $\sum b_n$.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ or if $a_n > b_n$ for all n, then $\sum a_n$ dominates $\sum b_n$.
- Remember: You don't get any information from domination if the smaller sum converges or the bigger sum diverges.

Absolute vs. Conditional Convergence

- A series $\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges.
- A series $\sum a_n$ *converges conditionally* if $\sum a_n$ converges, but $\sum |a_n|$ doesn't.

Ratio Test

- For any series $\sum a_n$, let $r = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. There are three cases:
  1. If $r < 1$, then $\sum a_n$ converges absolutely.
  2. If $r = 1$, the ratio test is inconclusive.
  3. If $r > 1$, then $\sum a_n$ diverges.