Infinite Sums **Math 421 - Workshop**

Functions can be expressed as infinite Taylor series (also known as power series). The two most useful Taylor series in probability theory are:

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ Converges for all x.
- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$ Only converges if $|x| < 1$.

Other Taylor series can be constructed from these by multiplying, differentiating, and integrating. Remember these linearity rules apply to all series:

$$\sum c\, a_n = c \sum a_n \quad \text{and} \quad \sum a_n + \sum b_n = \sum (a_n + b_n).$$

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1. Differentiate the power series for $\frac{1}{1-x}$ to get a power series for $\frac{1}{(1-x)^2}$.

2. Integrate the power series for $\frac{1}{1-x}$ to get a power series for $\ln(1-x)$.

3. Evaluate the sum: $\frac{1}{2} + \frac{1}{2(4)} + \frac{1}{3(8)} + \frac{1}{4(16)} + \frac{1}{5(32)} + \dots$.

4. Use the power series for $\frac{1}{1-x}$ to show that the sum of any geometric series with first term a and common ratio r is $\frac{a}{1-r}$.

5. Evaluate the sum: $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \dots$.

6. Differentiate the power series for e^x. Write out the first 5 terms of the derivative. What do you notice?

7. Suppose that $X \sim \text{Pois}(\lambda)$. Write $E(X)$ as an infinite series in both summation and term-by-term notation.

8. Use the power series for e^x to find $E(X)$ when X is a random variable with the $\text{Pois}(\lambda)$ distribution.

9. In class we saw that if $X \sim \text{Geom}(p)$, then the expected value of X is:

$$E(X) = \sum_{n=1}^{\infty} np(1-p)^{n-1}.$$

   Use your answer to Ex. 1 with $x = (1 - p)$ to find the expected value of X.

10. To find the variance of $X \sim \text{Pois}(\lambda)$, you need to find $E(X^2)$. Write this as an infinite sum using both summation and term-by-term notation.

11. Challenge: Try to evaluate the sum in the last problem. Your answer should be a function of λ.