Math 142 Group Projects

For this project you must work in groups of 2 or 3 students. You and your group will have to pick a topic by Thurdsay, November 16. No two groups may present the same topic and the topics are awarded on a first come first serve basis. Each group will have 15 minutes to present their topic one day during the last week of classes. Every member of the group must speak during the presentation and be ready to answer questions.

Euler's formula.

1. Define the complex number i.
2. Explain Euler's formula and show how it can be derived using the Maclaurin series for sine, cosine, and e^x.
3. Show that Euler's formula immediately implies Euler's identity $e^{i\pi} + 1 = 0$ which relates the five most important numbers in all of mathematics.
4. Show how Euler's formula implies de Moivre's formula $(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$, and draw a picture demonstrating how de Moivre's formula works on the unit circle in the complex plane.
5. Use Euler's formula to prove the angle addition identities for sine and cosine.

Archimedes and the quadrature of the parabola.

1. How did the ancient Greeks describe parabolas?
2. Give a description of how Archimedes was able to compute the area inside of a parabola without using integral calculus. Be sure to include details on how he added up an infinite geometric series, and where the terms in the series come from.
3. Use integral calculus to find the same area that Archimedes found.

e is irrational. One way to prove that e is an irrational number is to use a *proof by contradiction*. You start by assuming that e is rational, i.e., $e = \frac{m}{n}$ where m and n are positive integers. This assumption will lead to a logical contradiction, which means that it cannot be true. To get the contradiction, you'll need a second formula for e. By Taylor's theorem: $e = S_n + R_n$, where

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$$

is the n^{th} partial sum of the Maclaurin series for e and R_n is the remainder:

$$R_n = \frac{e^z}{(n+1)!} \quad \text{(for some } z \text{ between 0 and 1).}$$

1. Show that $n!S_n$ must be an integer.
2. Show that $0 < n!R_n < 1$.
3. Show that the two formulas $e = \frac{m}{n}$ and $e = S_n + R_n$ say different things about whether $n!e$ is an integer. What does this contradiction force us to conclude?
4. What other famous constants in math are irrational? Do some research about the history of this subject, and tell us a little of what you discover.

Gabriel's horn. Gabriel's horn is a shape that has a finite volume, but an infinite surface area. So if you completely filled it with paint, you still wouldn't have enough paint to cover the outside.

1. Read Section 8.3 in the book about how to find the surface area of a surface of revolution. Your first job is to teach the class how to calculate surface area for surfaces of revolution.
2. Gabriel's horn is the surface obtained by revolving the curve $y = 1/x$, with $1 \leq x < \infty$, around the x-axis. Show how to calculate the surface area of Gabriel's horn.
3. Find the volume of the region under the curve $y = 1/x$ with $1 \leq x < \infty$ when it's revolved around the x-axis. In other words, what is the volume inside Gabriel's horn?

The gamma function. The *gamma function* is a generalization of the factorial function that works for real numbers, not just whole numbers. It is defined by the integral $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$.

1. Find $\Gamma(1)$.
2. Use integration by parts to show that that $\Gamma(x) = (x-1)\Gamma(x-1)$.
3. Combine your solutions to parts 1 & 2 to find $\Gamma(2)$. Then find $\Gamma(3)$ and $\Gamma(4)$.
4. Explain why $\Gamma(n) = (n-1)!$ for every positive integer n.
5. Do a little research about the Gamma function. What are its applications? What does the graph of $\Gamma(x)$ look like?

Euler and the infinitude of primes. Euler devised an amazing proof that there are infinitely many prime numbers by considering the harmonic series.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

He observed that the harmonic series must be equal to the product

$$\prod_p \left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots\right)$$

taken over all prime numbers p. (Here the $\prod$ symbol works just like the $\sum$ symbol, except you multiply instead of add).

1. Try expanding the first few terms (say when $p = 2, 3$, and 5) of the product above to get a feel for how it works.
2. Use the fact that every integer $n > 1$ has a unique prime factorization to explain why the product formula above contains every single term in the harmonic series exactly once.
3. Explain why $\sum_{n=0}^{\infty} 1/p^n$ is finite for every prime number p.
4. How could Euler conclude that there must be infinitely many primes?

The 68-95-99.7 rule. The normal distribution (bell curve) is an important tool in statistics. The formula for the normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

1. Start by explaining the 68-95-99.7 rule from statistics (assume that not everyone in the class has seen it before).
2. It is impossible to integrate the bell curve explicitly using the functions we know, so you will demonstrate two different approximation techniques: Riemann sums and Maclaurin series. I recommend using Sage, but you could also use a spreadsheet if you prefer.
3. Use a Macluarin series to estimate the area under the normal distribution on the following three intervals: $[-1, 1]$, $[-2, 2]$, and $[-3, 3]$.
4. Compute a Riemann sum with a small Δx to estimate the area under the normal distribution on the intervals $[-1, 1], [-2, 2]$, and $[-3, 3]$. Which is more accurate, the Riemann sum with 100 rectangles or the Maclaurin series with 100 terms?

Binomial series. The binomial series was discovered by Isaac Newton around 1665.

1. Describe the binomial series by deriving the Maclaurin series for $(1 + x)^\alpha$. Be sure to explain the formula for the binomial coefficients $\binom{\alpha}{k}$ and find the radius of convergence.
2. Explain how the binomial series is different depending on whether or not α is a positive integer.
3. Explain how you can use Pascal's triangle to find the binomial coefficients $\binom{n}{k}$ when n is integer.
4. Show how to use the binomial series to estimate square roots by estimating $\sqrt{\frac{5}{4}}$ and $\sqrt{5}$.