Math 422 - Week 4 Homework **Due Monday, Feb 19**

We say that a collection of n random variables $(X_1, \ldots, X_n)$ has a **multivariate normal distribution** if they have a joint probability density function f defined for $x \in \mathbb{R}^n$ by

$$f(x) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\tfrac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right).$$

Here $\mu = E(X) = (E(X_1), \ldots, E(X_n))$ is the vector of expected values, and

$$\Sigma = \begin{bmatrix} \operatorname{Cov}(X_1, X_1) & \ldots & \operatorname{Cov}(X_1, X_n) \\ \vdots & \ddots & \vdots \\ \operatorname{Cov}(X_n, X_1) & \ldots & \operatorname{Cov}(X_n, X_n) \end{bmatrix}$$

is the covariance matrix.

Fact 1. Suppose that $X = (X_1, \ldots, X_n)$ has a multivariate normal (MVN) distribution with means $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ and covariance matrix Σ. Let $A \in \mathbb{R}^{m \times n}$ be any m-by-n matrix and $c \in \mathbb{R}^n$ be a vector. If Y is an affine linear transformation of X given by

$$Y = AX + c,$$

then Y has a multivariate normal distribution with means $\mu_Y = A\mu + c$ and covariance matrix $\Sigma_Y = A\Sigma A^T$.

Fact 2. Suppose $X = (X_1, \ldots, X_n)$ has a multivariate normal distribution. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be vectors in $\mathbb{R}^n$. Then

$$\operatorname{Cov}(a^TX, b^TX) = a^T\Sigma b$$

and the random variables a^TX and b^TX are independent if and only if $a^T\Sigma b = 0$.

Exercises

1. Suppose that $(X_1, X_2, \ldots, X_n)$ are i.i.d., normal random variables with mean μ and variance σ^2. What is the covariance matrix Σ for these variables?

2. Suppose that X_1, X_2, X_3 are independent, Normal$(0,1)$ random variables. Find the covariance matrix and joint density function for Y_1, Y_2, and Y_3 given by:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

3. Suppose that $X_1, \ldots, X_n$ have a MVN distribution with means μ and covariance matrix Σ. Show that any subset of the variables $(X_1, \ldots, X_n)$ also has a MVN distribution.

4. Prove Fact 2 using Fact 1.

5. Let $\bar{x} = \frac{1}{n}(X_1 + \ldots + X_n)$ where $X_1, \ldots, X_n$ are i.i.d. Normal random variables. Show that $\bar{x}$ and $X_1 - \bar{x}$ are independent.