Math 422 - Week 5 Homework **Due Wednesday, Mar 14**

For sample data $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, the correlation between x and y is

$$r = \frac{(x - \bar{x})^T (y - \bar{y})}{\|x - \bar{x}\| \|y - \bar{y}\|}$$

The least squares solution of an inconsistent linear equation $X\beta = y$ is given by solving $(X^T X)\hat{\beta} = X^T y$. If the columns of X are linearly independent, then $X^T X$ is invertible and

$$\hat{\beta} = (X^T X)^{-1} X^T y.$$

Problems

1. Use a computer to find the least squares solutions of $X\beta = y$ for the following matrices.

   (a) $X = \begin{bmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{bmatrix}$, $y = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

   (b) $X = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$, $y = \begin{bmatrix} 5 \\ -3 \\ 5 \end{bmatrix}$

   (c) $X = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$, $y = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$

2. Suppose that $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are points in a scatterplot. Use the formula for correlation above to show that if all of these points are on a line $y = mx + b$ with positive slope, then the correlation is one. Hint: if you think of x and y as vectors, then $y = mx + b\mathbb{1}$. Also, $\bar{x} = \frac{1}{n} x^T \mathbb{1}$.

3. Suppose an object is thrown off of a tall cliff (200 meters tall) and the elevation of the object above the ground is measured each second as it falls (see table below). Find the coefficients β_i of the parabola that best predicts the height of the object as a function of time $y = \beta_0 + \beta_1 t + \beta_2 t^2$. In particular, what is the initial vertical velocity and the acceleration of the object?

Time (seconds)	Elevation (meters)
0	200.0
1	205.3
2	200.2
3	185.5
4	161.5
5	127.3
6	83.5
7	29.8