Math 422 - Homework 9 **Due Monday, April 30**

1. Suppose that whether or not it rains tomorrow only depends the last two days. If it rained both today and yesterday, then there is an 80% chance that it will rain tomorrow. If it rained today, but not yesterday, then there is a 50% chance that it will rain tomorrow. If it rained yesterday, but not today, then there is a 10% chance that it will rain tomorrow. Finally, if it did not rain either today or yesterday, then there is a 20% chance that it will rain tomorrow.

   (a) Draw a graph showing the states and the transition probabilities for a Markov chain that models this situation. *Hint: The states in this Markov chain are the weather over two consecutive days, not just the weather on one day.*

   (b) Is this Markov chain irreducible? How can you tell?

   (c) Over time, in what percent of days does it rain?

2. Show that if the transition matrix P for a Markov chain is symmetric (i.e., $P^T = P$), then the vector $(\frac{1}{n}, \ldots, \frac{1}{n})$ is a stationary probability vector for the Markov chain.

3. Suppose that a car rental company has four locations (called them locations 1, 2, 3, and 4). Rented cars can be returned to any of the four locations. Suppose that the probability a car that is rented from location j is returned to location i is given by the entries p_{ij} of the matrix:

$$P = \begin{bmatrix} 0.8 & 0.1 & 0.2 & 0 \\ 0.1 & 0.7 & 0.1 & 0.2 \\ 0 & 0.2 & 0.5 & 0.1 \\ 0.1 & 0 & 0.2 & 0.7 \end{bmatrix}$$

   (a) A car is currently at location 3. What is the probability that this car will be back at location 3 after it is rented 5 times?

   (b) After a car has been rented many many times, what is the probability distribution for its location?

4. The *birthday problem* is a famous problem in probability theory. It asks: "How many people need to be in a room before there are even odds that at least two people in the room have the same birthday?" You can use a Markov chain to model the birthday problem. Let the state of the Markov chain be the number of different birthdays that a group of people in a room have. When a new person enters the room, the state will transition to a higher value if the new person has a birthday that is different from everyone else, otherwise it will stay the same. Because the state can be anywhere from 1 to 365 (if you ignore leap years), the transition matrix is too big to write down on paper. Instead, find a formula for the entry p_{ij} in row i and column j of the transition matrix, as a (piecewise) function of i and j.