Introduction to Bayesian Statistics **Math 422 - Workshop**

Suppose I have a box with 3 balls in it. Some of the balls are red and the rest are black. Let N be the parameter that represents the number of red balls in the box.

1. If you reach into the box, and the first ball you randomly select is red, then what is the likelihood function for N?

2. Before randomly selecting a ball, suppose you had no idea what the value of N was. One way to mathematically model your uncertainty is to assume that N is equally likely to be 0, 1, 2, or 3. Make a tree diagram showing the possible values of N and then the possible outcomes (red or black) and use it to find the conditional probability that $N = 3$ given that the first ball was red.

3. Another way to calculate the conditional probability from the last problem is to use **Bayes' Theorem** which says:

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(B)}.$$

In the last problem, we used a tree diagram to find $P(N = 3 \mid \text{Ball was red})$. Now, use the Bayes' theorem formula to find the probability that N is each of the other three possible values (0, 1, or 2), given that the first ball was red. That is, calculate

$$P(N = k \mid \text{Ball was red}) = \frac{P(N = k) \cdot P(\text{Ball was red} \mid N = k)}{P(\text{Ball was red})} \qquad \text{for } k = 0, 1, 2.$$

Before we drew a ball from the box, we didn't know the value of N. We used a simple discrete uniform distribution to model our uncertainty about N. We call this initial probability distribution our **prior distribution**. Once we got a red ball, we were able to calculate a new probability distribution for N called the **posterior distribution**. The posterior distribution tells us everything that we know about N after we draw a ball from the box. To get the posterior distribution, we could use Bayes' formula or a tree diagram, but there is a faster way: just multiply the prior distribution by the likelihood function for each value of N. Then your posterior distribution is just a scalar multiple of the result (scale by whatever constant you need to make the total probability equal to one). This formula summarizes the idea (the symbol $\propto$ means "is proportional to"):

$$\text{posterior} \propto \text{prior} \times \text{likelihood}$$

4. Complete the table below to find the posterior distribution for N a different way.

N	prior	likelihood	prior $\times$ likelihood	posterior
0	0.25			
1	0.25			
2	0.25			
3	0.25			
Total	1			

5. Suppose a box has 7 balls and N of them are red. You don't know the value of N so you start with a uniform prior distribution. In order to find N, you draw a sample of 3 balls from the box. If 2 of the balls in the sample are red, then what is the posterior distribution for N? Note: in this example you will need to use the hypergeometric distribution to find the likelihood function.