Bayesian Statistics - Part 2 **Math 422 - Workshop**

Suppose that $X \sim \text{Binom}(5, p)$. For example, maybe X represents a random sample of 5 observations from a large population, and p is the probability of a "success".

1. What is the likelihood function for p if $X = 4$?

2. If we start with a Unif(0, 1) prior distribution for p, then the posterior distribution will be proportional to the likelihood function on the interval from $p = 0$ to $p = 1$. What is the integral of the likelihood function on this interval?

3. Show that the posterior distribution for p given that $X = 4$ is $30\,p^4(1-p)$. That is, show that the posterior distribution is proportional to $p^4(1-p)$ and the right proportionality constant is 30.

Notation: For any unknown parameter θ, we will use $\pi(\theta)$ to represent the prior distribution and $\pi(\theta \,|\, s)$ to represent the posterior distribution for θ given a sample s. So the posterior distribution in the example above is would be written like this:

$$\pi(p \,|\, X = 4) = \begin{cases} 30\,p^4(1-p) & p \in 0 \le p \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

This probability distribution is an example of a **beta distribution**. The PDF for a beta distribution has two (positive) parameters α and β, and the formula is:

$$f(x) = Cx^{\alpha-1}(1-x)^{\beta-1}$$

when $x \in [0, 1]$, and zero otherwise. The constant C is whatever value makes the area under the curve equal to one. In the example above: $\alpha = 5$ and $\beta = 2$.

4. Use Desmos to graph the beta distribution for different values of α and β. What happens to the shape if α and β are equal? What happens to the shape as α and β both get bigger? What happens to the shape if only one of α or β is big?

5. Beta distributions are very useful in Bayesian statistics because if p is the unknown parameter for a binomial distribution and you start with a beta distribution as your prior, you always end up with a beta distribution for your posterior. Show this. Suppose $X \sim \text{Binom}(n, p)$ for some fixed n and unknown p. Suppose we start with a prior distribution $\pi(p) \propto p^{\alpha-1}(1-p)^{\beta-1}$. Show that the posterior distribution $\pi(p \mid X = k)$ has a beta distribution no matter what k is.