Gamma Distribution **Math 422 - Workshop**

Another probability distribution that is useful in Bayesian statistics is the **Gamma distribution**. It is a continuous probability distribution on $[0, \infty)$ with two parameters α and λ (which are both positive real numbers). The density function for the Gamma(α, λ) distribution is:

$$f(x) = \frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x}, \text{ for } x \geq 0$$

where $\Gamma(\alpha)$ is called the **Gamma function** and is equal to whatever number makes the integral of the density function come out to one. It is not hard to show (using a u-substitution) that the Gamma function is

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx.$$

1. (Optional) Use integration by parts to prove the following very important fact about the Gamma function: for any $\alpha > 0$,
$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$$

2. (Optional) Give a proof by induction that $\Gamma(n) = (n-1)!$ for every positive integer n.

3. It is a fact that $\Gamma(1/2) = \sqrt{\pi}$. Without integrating, what are $\Gamma(3/2)$ and $\Gamma(5/2)$?

You've already seen some examples of Gamma distributions before. The exponential distribution Exp(λ) is just the Gamma distribution with $\alpha = 1$. The Chisq(n) distribution is also a special case of the Gamma distribution, with $\alpha = \frac{n}{2}$ and $\lambda = \frac{1}{2}$.

Now we'll look at two examples of how to apply the Gamma distribution in Bayesian statistics.

4. Suppose that $x_1, \ldots, x_n$ is a sample from a $\text{Pois}(\lambda)$ distribution with $\lambda \geq 0$ unknown. If we use $\text{Gamma}(\alpha, \beta)$ as the prior distribution for λ, then what is the posterior distribution for λ given our sample data?

5. Suppose that $x_1, \ldots, x_n$ is a sample from a $\text{Unif}(0, N)$ distribution with $N > 0$ unknown. If we use $\text{Gamma}(\alpha, \beta)$ as the prior distribution for N, then what is the posterior distribution for N given our sample data?