Likelihood Functions **Math 422 - Workshop**

The probability mass function (PMF) for a Poisson distribution with parameter λ is:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

We think of λ as a constant and k as the input variable. But what if we know k, and don't know λ? When we treat k as a constant and λ as a variable, we get the **Likelihood function**:

$$L(\lambda \,|\, X = k) = \frac{e^{-\lambda}\lambda^k}{k!}.$$

Intuitively, $L(\lambda \,|\, X = k)$ is the likelihood that parameter λ has a particular value, given the outcome for X that we saw. It is a function of λ, and sometimes we just write $L(\lambda \,|\, X)$ or even just $L(\lambda)$ to emphasize this. You create likelihood functions for continuous random variables the same way, except using probability density functions (PDFs) instead of PMFs.

1. Some geologists set up monitoring equipment to find out how often a small island has major earthquakes. Let X be the time until they detect the first major earthquake and assume that $X \sim \text{Exp}(\lambda)$ so the PDF is: $f_X(x) = \lambda e^{-\lambda x}$.

   (a) If it takes $X = 4$ years until the first major earthquake, then what is the likelihood function $L(\lambda \,|\, X = 4)$?

   (b) **Important:** The likelihood function is not the probability distribution for λ. But you can use it to say that certain values of the parameter λ are more likely than others, given our observation that $X = 4$. How many times more likely is it that $\lambda = \frac{1}{5}$ than that $\lambda = 1$ according to the likelihood function above?

   (c) Take the derivative of the likelihood function in part (a) to find the value of λ with the maximum likelihood. Then graph the likelihood function and see if you got the right answer.

2. You visit a large university and are wondering how many of the students are science majors. You pick one random student and find out that she is a science major. Let X be the random variable representing whether a student is a science major (so the values that X can take are either 1 if a student is a science major, or 0 if they are not). Assume that the true proportion of students on campus who are science majors is p.

   (a) What is the probability distribution for X and what is its probability mass function?

   (b) What is the likelihood function for the parameter p given that $x = 1$?

3. What if, instead of interviewing one student, you took a random sample of six students and 2 of the 6 were science majors. You can think of the number of science majors in the sample a s a random variable with the binomial distribution $\text{Binom}(6, p)$.

   (a) What is the likelihood function for p given that there were 2 science majors in our sample?

   (b) Show that the maximum value of the likelihood function $L(p \mid x = 2)$ occurs when $p = \frac{1}{3}$. Hint: it is often easier to maximize the log-likelihood function, which is just the natural logarithm of $L(p)$.