**Maximum Likelihood Estimation - Part 2** **Math 422 - Workshop**

Last time we looked at maximum likelihood estimation (MLE) with one unknown parameter. Today, we will look at an important example with more than one unknown parameter. Suppose that we have a large population that can be modeled by a normal distribution with mean $\mu$ and standard deviation $\sigma$. Statisticians often use the symbol $\theta$ to represent all of the parameters of a model. In this case, $\theta$ is the vector $(\mu, \sigma)$. Our goal is to find the MLE for $\theta$ given a set of $n$ observations $x_1, \ldots, x_n$.

1. The likelihood function for $\theta$ is:

$$L(\theta \,|\, x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}\sigma^n} \exp\left( -\frac{(x_1 - \mu)^2}{2\sigma^2} - \ldots - \frac{(x_n - \mu)^2}{2\sigma^2} \right).$$

   Find the log-likelihood function and simplify as much as you can.

2. Find the partial derivatives of the log-likelihood function with respect to $\mu$ and with respect to $\sigma$.

3. Set both partial derivatives in the last problem equal to zero and solve the two equations for the two unknowns $\mu$ and $\sigma$. Hint: find a formula for $\mu$ first using the simpler equation, then substitute into the other equation to get $\sigma$.

4. **(Optional Challenge Problem)** In the last question, you found the critical point of the log-likelihood function, but we still have to double check that it is a maximum. To do this, you need to find the Hessian matrix (the matrix of 2nd derivatives):

$$H = \begin{bmatrix} \dfrac{\partial^2 \log L}{\partial \mu^2} & \dfrac{\partial^2 \log L}{\partial \mu \partial \sigma} \\[2ex] \dfrac{\partial^2 \log L}{\partial \mu \partial \sigma} & \dfrac{\partial^2 \log L}{\partial \sigma^2} \end{bmatrix}$$

If both eigenvalues are negative, then the critical point is a local maximum (that's the 2nd derivative test in multivariable calculus). In this case, your Hessian matrix should be a diagonal matrix, so the eigenvalues will be the two diagonal entries. Find the Hessian and simplify to verify that both diagonal entries are negative at the critical point.