**Math 444 - Complex Functions** **Name:** ______________________

We have already defined $e^z = e^x(\cos y + i \sin y)$ for any $z = x + iy \in \mathbb{C}$. Now, if we solve that equation for cosine or sine we get two new equations:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

1. Show that $\frac{d}{dz} \sin z = \cos z$ and $\frac{d}{dz} \cos z = -\sin z$ for all $z \in \mathbb{C}$. Hint: You don't need to use the definition of the derivative or the Jacobian. Why not?

2. When $z$ is a real number, $\cos(z)$ and $\sin(z)$ are always bounded between 1 and $-1$. This isn't true for complex numbers. Find a formula for $\sin(iy)$ for any real number $y$ and then show that $\lim_{y \to \infty} \sin(iy) = \infty$.

3. Find all solutions of the equation $\sin z = 2$. Hint: Start by letting $u = e^{iz}$. Then $\sin z = \frac{u - u^{-1}}{2i} = 2$. If you multiply both sides of this equation by $2iu$, then you get a quadratic polynomial.

4. The angle addition formulas in trigonometry are:

$$\begin{aligned}\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b)\\ \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b)\end{aligned}$$

   Derive these for $a, b \in \mathbb{R}$ by multiplying $e^{ia}e^{ib}$.

5. It is a little tedious (but not hard) to prove that the angle addition formulas work for complex numbers. So let's just assume that they do. What happens if you take the derivative of both sides of the formula for $\sin(a+b)$ with respect to $a$?

*Convert the following to rectangular form.*

6. $e^{\sin(i)}$

7. $\mathrm{Log}(1+\sqrt{3}i)$

8. $\mathrm{Log}(\frac{1}{3+4i})$

9. One of the most important properties of the real logarithm is that $\ln(ab) = \ln(a) + \ln(b)$ for any positive real numbers $a, b$. Show that this property fails for the principal logarithm $\mathrm{Log}(z)$ by finding two complex numbers $a, b$ such that $\mathrm{Log}(ab) \neq \mathrm{Log}(a) + \mathrm{Log}(b)$.