Math 444 - Complex Limits **Name:** ______________________

- A **sequence** is a function $s : \mathbb{N} \to \mathbb{C}$. Notation: We usually write s_n instead of $s(n)$ for sequences.
- A sequence s_n **converges** to a point $z \in C$ if for every open disk centered at z, there are only finitely many values of n for which s_n is outside the disk. In other words, for every $\epsilon > 0$, there is a value N large enough so that $|s_n - z| < \epsilon$ for every $n \geq N$.
- For a function $f : \mathbb{C} \to \mathbb{C}$, we say that the **limit** of f as z approaches z_0 is L, that is,

$$\lim_{z \to z_0} f(z) = L$$

  if and only if $f(s_n)$ converges to L for every sequence s_n that both converges to z and is never exactly equal to z.
- A function $f : \mathbb{C} \to \mathbb{C}$ is **continuous** at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$. We also say f is **continuous on a set** G if f is continuous at every point in G.

1. Show that the sequence $s_k = \frac{1}{\sqrt{k}} e^{i \frac{k\pi}{6}}$ converges to 0. For any fixed $\epsilon > 0$ find a formula for an N that is big enough so that $|s_k - 0| < \epsilon$ whenever $k \geq N$. Your formula for N such be a function of ϵ.

2. If z_k is a sequence that converges to $a \in \mathbb{C}$, and $c \in \mathbb{C}$ is a constant, show that $c\, z_k$ converges to $c\, a$.

3. One method to calculate a complex limit $\lim_{z \to z_0} f(z)$ is to make the substitution $z = z_0 + re^{i\theta}$ and calculate $f(z_0 + e^{i\theta})$. If the expression does not depend on θ as $r \to 0$, then you can find the limit by letting $r = 0$. Try this on the function $f(z) = z^2$.

4. Let $f(z) = \frac{\overline{z}}{z}$. Show that the limit $\lim_{z\to 0} f(z)$ does not exist. Hint: Try the substitution from the previous problem. What goes wrong?

5. For any complex numbers $a, b, w, z \in \mathbb{C}$ prove that $|(z+w)-(a+b)| \leq |z-a| + |w-b|$.

6. Suppose that z_k is a sequence in $\mathbb{C}$ that converges to a and w_k is a sequence that converges to b. Use the definition of convergence to show that $z_k + w_k$ converges to $a+b$. Hint: By the definition of convergence, for any $\epsilon > 0$, there is an M such that $|z_k - a| < \epsilon/2$ when $k \geq M$. Likewise, there is an N such that $|w_k - b| < \epsilon/2$ when $k \geq N$.

7. Suppose that z_k is a sequence in $\mathbb{C}$ that converges to a nonzero complex number a. Prove that there is an N large enough so that $\frac{1}{2}|a| < |z_k|$ for all $k \geq N$.