Math 345: Applied Mathematics
VII: Noise

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Noise is unwanted signal. Often it is random in nature, like the background din of a busy city street. Sometimes it has more structure. One man's signal can be another man's noise, as for example the loudmouth at the table next to you in a nice restaurant. In such cases we usually call it *interference* or *bad manners*, rather than noise.

Separating signal from noise is one of the primary goals of signal processing. In fact, signal processing may be defined as the search for optimal strategies in the eternal battle between signal and noise. *Thermal noise* is one of the most pervasive types of noise to be dealt with in signal processing. Thermal noise consists of the random electrical signals generated by heat energy in a conductor. It is present in every electronic device. The simplest mathematical model for thermal noise is *additive white Gaussian noise* (AWGN).

Additive White Gaussian Noise. AWGN is an idealized conceptual model for thermal noise. Let $\sigma > 0$. An *AWGN waveform with variance* σ^2 is any signal $w(t)$ with the property that, for any positive integer n, and any sample times $t_1, t_2, \ldots, t_n$, the samples

$$\{w(t_1), w(t_2), \ldots, w(t_n)\}$$

are independent normal random variables, all with mean zero and standard deviation σ. While easy enough to define, AWGN has some very unusual properties. For instance, $w(t)$ is nowhere continuous, and is unbounded on any interval. A completely rigorous mathematical treatment of AWGN is highly nontrivial. But the above definition gets across the central idea. AWGN is important not only in signal processing, but in many other fields as well.[1]

Despite the mathematical difficulties inherent in AWGN, it is extremely easy to simulate: you simply need to generate independent Gaussian samples. In MATLAB®, the `randn` function provides this capability.

It is natural to wonder about the Fourier transform of AWGN. Since AWGN itself is random, its Fourier transform is also random. This makes $W = \mathbb{F}w$ of somewhat limited usefulness. Still, it would be nice to have some *non-random* indicator of how much power a random signal has in a given frequency band *on average*. The *power spectral density* function (PSD) does this. For any random signal s, the PSD is defined by

$$\text{PSD}[s](f) = \lim_{T\to\infty} \frac{1}{T} E\left(\left|\int_{-T/2}^{T/2} s(t)e^{-2\pi i f t}\,dt\right|^2\right). \tag{1}$$

The power of a signal in a given band is equal to the integral of the PSD over the band (both positive and negative frequencies):

$$\text{Power of } s \text{ between frequencies } f_0 \text{ and } f_1 = \int_{f_0}^{f_1} \text{PSD}[s](f)\,df + \int_{-f_1}^{-f_0} \text{PSD}[s](f)$$

[1] For instance, AWGN is related to *Brownian motion*, the famous random process modeling (among other things) the motion of particles suspended in a fluid and the fluctuations of prices in a market. In fact, Brownian motion $B(t)$ may be represented as

$$B(t) = \int_0^t dw(s)$$

where w is AWGN. The *Itō calculus* provides a rigorous development of these concepts.

If $s = w$ is AWGN, then it can be shown that the power spectral density is constant, and equal to the one half the variance σ^2:

$$\text{PSD}[w](f) = \frac{\sigma^2}{2} \quad \text{for all } f \tag{2}$$

This says that AWGN has *infinite bandwidth*, and that the power in any band is equal to σ^2 times the bandwidth. Note that the total power of AWGN (over all frequencies) is infinite.

Filtered Gaussian Noise. Because of its infinite bandwidth and infinite power, AWGN is not a good model for the noise actually encountered in signal processing. Actual noise is always of finite power and bandwidth. We can easily remedy this situation by considering *filtered Gaussian noise*. Filtered noise is the result of passing an AWGN signal through a filter. Filtered noise is sometimes referred to as *colored noise*. Suppose $H(f)$ is the transfer function of our filter, and $h = \mathbb{F}^{-1}H$ is its impulse response. If we pass a noise signal s with power spectral density PSD$[s]$ through the filter, it can be shown that the filtered signal has a power spectral density function that is the product of the original PSD with the transfer function:

$$\text{PSD}[s * h](f) = \text{PSD}[s](f) \cdot H(f). \tag{3}$$

Of particular interest is *band-limited Gaussian noise*, which is the result of passing AWGN through a bandpass filter.

Exercise 1. Write a simple MATLAB®function to generate AWGN. The function should take as inputs the variance, duration, and sampling rate. The output of the function should be an array containing the AWGN samples.

Exercise 2. Use the function from Exercise 1 and our `sBandPassFilter` to simulate band-limited Gaussian noise.

Exercise 3. Use MATLAB®to simulate band-limited Gaussian noise with the following parameters:

1. $\sigma^2 = 1$, $f_{\text{low}} = 0$ hz, $f_{\text{high}} = 500$ hz, duration = 3 seconds.
2. $\sigma^2 = 1$, $f_{\text{low}} = 200$ hz, $f_{\text{high}} = 500$ hz, duration = 3 seconds.
3. $\sigma^2 = 1$, $f_{\text{low}} = 400$ hz, $f_{\text{high}} = 500$ hz, duration = 3 seconds.
4. $\sigma^2 = 1$, $f_{\text{low}} = 450$ hz, $f_{\text{high}} = 500$ hz, duration = 3 seconds.
5. $\sigma^2 = 1$, $f_{\text{low}} = 490$ hz, $f_{\text{high}} = 500$ hz, duration = 3 seconds.

Use a sampling rate of $f_s = 11025$ hertz for all simulations. In each case, make two graphs the noise signal in the time domain, one from $t = 0$ to $t = 3$ seconds, and the other from $t = 0$ to $t = 0.1$ seconds. Also, graph the magnitude of the Fourier transform of the noise signal. Finally, use MATLAB®'s `sound` function to listen to the noise waveforms. Describe what you find in a couple of well-written paragraphs.