Closure Properties of Decidable and Recognizable Languages

Lecture 28
Problems 3.15 and 3.16

Robb T. Koether
Hampden-Sydney College
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Exercise 3.6, page 160.

In Theorem 3.21 we showed that a language is Turing-recognizable iff some enumerator enumerates it. Why didn’t we use the following simpler algorithm for the forward direction of the proof? As before, $s_1, s_2, \ldots$ is a list of all strings in $\Sigma^*$.

$E = “$Ignore the input.$”$

1. Repeat the following for $i = 1, 2, 3, \ldots$

2. Run $M$ on $s_i$.

3. If it accepts, print out $s_i.”$
Exercise 3.7, page 160.

Explain why the following is not a description of a legitimate Turing machine.

\( M_{\text{bad}} = \text{“The input is a polynomial } p \text{ over variables } x_1, \ldots, x_n. \)

1. Try all possible settings of \( x_1, \ldots, x_n \) to integer values.
2. Evaluate \( p \) on all of these settings.
3. If any of these settings evaluates to 0, accept; otherwise, reject.”
Theorem (Closure Properties of Decidable Languages)

The class of decidable languages is closed under

- Union
- Intersection
- Complementation
- Concatenation
- Star
Closure Under Intersection

**Theorem**

*If $L_1$ and $L_2$ are decidable, then $L_1 \cap L_2$ is decidable.*
Proof.

- Let $D_1$ be a decider for $L_1$ and let $D_2$ be a decider for $L_2$.
- Then build a decider $D$ for $L_1 \cap L_2$ as in the following diagram.
Theorem

If $L_1$ and $L_2$ are decidable, then $L_1 \cup L_2$ is decidable.
Proof.

Let $D_1$ be a decider for $L_1$ and let $D_2$ be a decider for $L_2$.

Then build a decider $D$ for $L_1 \cup L_2$ as in the following diagram.

![Diagram](image-url)
Closure Under Other Operators

- How would we show that if $L_1$ and $L_2$ are decidable, then so are
  - $L_1 L_2$
  - $\overline{L_1}$
  - $L_1^*$
Theorem (Closure Properties of Recognizable Languages)

The class of recognizable languages is closed under

- Union
- Intersection
- Concatenation
- Star
Closure Under Intersection

**Theorem**

If $L_1$ and $L_2$ are recognizable, then $L_1 \cap L_2$ is recognizable.
Closure Under Intersection

Proof.

- Let $R_1$ be a recognizer for $L_1$ and let $R_2$ be a recognizer for $L_2$.
- Then build a recognizer $R$ for $L_1 \cap L_2$ as in the following diagram.

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R
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\begin{center}
\begin{tikzpicture}
\node (R1) at (0,0) {$R_1$};
\node (R2) at (1,0) {$R_2$};
\node (R) at (0.5,0) {$R$};
\node (w) at (-1,0) {$w$};
\draw[->] (w) -- (R1) node[midway, above] {yes};
\draw[->] (R1) -- (R2) node[midway, above] {yes};
\draw[->] (R2) -- (R) node[midway, above] {yes};
\end{tikzpicture}
\end{center}
```
Closure Under Union

Theorem

If $L_1$ and $L_2$ are recognizable, then $L_1 \cup L_2$ is recognizable.
Proof.

- Let $R_1$ be a recognizer for $L_1$ and let $R_2$ be a recognizer for $L_2$.
- Then build a recognizer $R$ for $L_1 \cup L_2$ as in the following diagram.
Closure of Union

Proof.

- In that diagram, we must be careful to alternate execution between $R_1$ and $R_2$. 
Closure Under Other Operators

- How would we show that if $L_1$ and $L_2$ are recognizable, then so are
  - $L_1 L_2$
  - $L_1^*$
- Why is $\overline{L_1}$ not necessarily recognizable?
Homework

- Read Section 3.2, pages 152 - 154.
- Problems 15, 16, page 161.