A Turing-Unrecognizable Language

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Outline

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2. $A_{TM}$ is Undecidable
   - The Turing Machine $H$
   - The Turing Machine $D$

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Exercise 4.24, page 184.

Let

\[ \text{PAL}_{\text{DFA}} = \{ \langle M \rangle \mid M \text{ is a DFA that accepts some palindrome} \} \].

Show that \( \text{PAL}_{\text{DFA}} \) is decidable. (Hint: Theorems about CFLs are helpful here.)
Theorem

$A_{TM}$ is undecidable.
Proof.

- Suppose that $A_{TM}$ is decidable.
- Then there is a Turing machine $H$ that decides $A_{TM}$.
- Thus, when $H$ reads $\langle M, w \rangle$, it will
  - Halt in its accept state if $M$ accepts $w$.
  - Halt in its reject state if $M$ does not accept $w$ (rejects $w$ or loops).
The Turing Machine $H$

Proof.

$H$

$M$ accepts $w$

yes

$M$ does not accept $w$

no

$\langle M, w \rangle$
$A_{TM}$ is Undecidable

**Proof.**

- We will use $H$ as a module in a new Turing machine $D$.
- $D$ will read a description $\langle M \rangle$ of a Turing machine $M$.
- Then $D$ will run $H$ on input $\langle M, \langle M \rangle \rangle$. 
$A_{TM}$ is Undecidable

**Proof.**

- If $H$ determines that $M$ accepts on $\langle M \rangle$, then $D$ will reject $\langle M \rangle$.
- If $H$ determines that $M$ rejects on $\langle M \rangle$, then $D$ will reject $\langle M \rangle$.
- That is, $D$ accepts $\langle M \rangle$ if and only if $M$ rejects $\langle M \rangle$. 
The Turing Machine D

Proof.

\[ \langle M \rangle \quad \langle M, \langle M \rangle \rangle \quad H \]

- **yes**
- **no**

\[ D \]

The Turing Machine D is Undecidable.
Proof.

- Now, run $D$ on $\langle D \rangle$.
- What happens?
The Turing Machine D

Proof.

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Assignment

\[ D \]

\[ \langle D \rangle \]

\[ \langle D, \langle D \rangle \rangle \]

\[ H \]

yes

no

yes

no

\[ D, \langle D \rangle \]
\( A_{TM} \) is Undecidable

Proof.

- According to the description of \( D \), \( D \) accepts \( \langle D \rangle \) if and only if \( D \) rejects \( \langle D \rangle \).
- This is a contradiction.
- Therefore, \( A_{TM} \) is undecidable.
The Halting Problem - Undecidability

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Homework Review

A Turing-Unrecognizable Language

**Theorem**

The language $\overline{A_{TM}}$ is Turing-unrecognizable.

**Lemma**

A language $L$ is decidable if and only if $L$ and $\overline{L}$ are Turing-recognizable.
Proof of the lemma ($\Rightarrow$).

- If $L$ is decidable, then there is a Turing machine $M_1$ that decides it.
- Then obviously $M_1$ recognizes $L$.
- To create a machine $M_2$ that recognizes $\overline{L}$, simply swap the accept and reject states of $M_1$. 
Proof of the lemma ($\iff$).

- Suppose $L$ and $\overline{L}$ are Turing-recognizable.
- Then there is a Turing machine $M_1$ that recognizes $L$ and a Turing machine $M_2$ that recognizes $\overline{L}$.
- We will build a Turing machine $D$ that decides $L$. 
Proof of the lemma ($\iff$).

- Given an input string $w$,
  - $D$ will run $M_1$ on $w$ for 1 step.
  - Then $D$ will run $M_2$ on $w$ for 1 step.
  - Then $D$ will run $M_1$ on $w$ for 2 steps.
  - Then $D$ will run $M_2$ on $w$ for 2 steps.
  - And so on.

- Since either $w \in L$ or $w \in \overline{L}$, eventually one of the machines will halt in its accept state.
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Proof of the lemma ($\iff$).

- When either one halts, $D$ halts.
- If $M_1$ halts in its accept state, then $D$ accepts $w$.
- If $M_2$ halts in its accept state, then $D$ rejects $w$.
- Thus, $D$ decides $L$. 
A Turing-Unrecognizable Language

Proof of the theorem.

- Suppose $\overline{A_{TM}}$ were Turing-recognizable.
- We already know that $A_{TM}$ is Turing-recognizable.
- It would follow from the lemma that $A_{TM}$ is decidable, which it is not.
- Therefore, $\overline{A_{TM}}$ is not Turing-recognizable.
This means that it is impossible to write a computer program that can read the code of any computer program and an input and decide whether that program will halt on that input.
Homework Review
$A_{TM}$ is Undecidable
The Turing Machine $H$
The Turing Machine $D$
A Turing-Unrecognizable Language
Assignment

Homework

- Read Section 4.2, pages 179 - 182.