Outline

1. Homework Review
2. The Regular Operations
   - Examples
3. Closure Properties
4. Assignment
Exercise 1.2

Give a formal description of the machine $M_1$ pictured below.
The set of states is $Q = \{q_1, q_2, q_3\}$.
The input alphabet is $\Sigma = \{a, b\}$.
The start state is $q_1$.
The set of final states is $F = \{q_2\}$.
The transition function is

$$\delta = \{(q_1, a, q_2), (q_1, b, q_1), (q_2, a, q_3),$$
$$ (q_2, b, q_3), (q_3, a, q_2), (q_3, b, q_1)\}.$$
The Regular Operations

Definition (Union of languages)
The union of languages \( A \) and \( B \) is the language

\[
A \cup B = \{ w \mid w \in A \text{ or } w \in B \}.
\]

Definition (Concatenation of languages)
The concatenation of languages \( A \) and \( B \) is the language

\[
A \circ B = \{ uv \mid u \in A \text{ and } v \in B \}.
\]

Definition (Kleene star of a language)
The Kleene star of a languages \( A \) is the language

\[
A^* = \{ w_1w_2 \ldots w_k \mid w_i \in A \text{ and } k \geq 0 \}.
\]
The Regular Operations

- We often abbreviate $A \circ B$ as $AB$.
- The Kleene star of $A$ can be written as

$$A^* = \{ \epsilon \} \cup A \cup AA \cup AAA \cup \cdots.$$
Examples (Regular operations)

- Let $A = \{w \mid w \text{ contains an even number of } a\text{'s}\}$.
- Let $B = \{w \mid w \text{ contains an even number of } b\text{'s}\}$.
- Describe the languages
  - $A \cup B$
  - $A \circ B$
  - $A^*$
  - $(A \cup B)^*$
  - $(A \circ B)^*$
  - $(A^*)^*$
Examples (Regular operations)

Design finite automata that accept

- $A \cup B$
- $A \circ B$
- $A^*$
- $(A \cup B)^*$
- $(A \circ B)^*$
- $(A^*)^*$
Examples (Regular operations)

- A DFA for $A \cup B$. 
Examples (Regular operations)

- A DFA for $A \circ B$. 

\[
\begin{array}{c}
\text{a, b} \\
\text{a} \\
\text{b} \\
\text{b} \\
\end{array}
\]
Theorem (Closure of Regular Languages)

The class of regular languages is closed under the operations of union, concatenation, and star.
Closure

Proof.

Proof (union)

- Let $M_1 = \{Q_1, \Sigma_1, \delta_1, q_1, F_1\}$ be a DFA whose language is $L_1$.
- Let $M_2 = \{Q_2, \Sigma_2, \delta_2, q_2, F_2\}$ be a DFA whose language is $L_2$.
- We will define a DFA $M$ whose language is $L_1 \cup L_2$.
- Let $M = \{Q, \Sigma, \delta, q_0, F\}$ where
  - $Q = Q_1 \times Q_2$.
  - $\Sigma = \Sigma_1 \cup \Sigma_2$.
  - $q_0 = (q_1, q_2)$.
  - $F = \{(p_1, p_2) \mid p_1 \in F_1 \text{ or } p_2 \in F_2\}$.
Proof.

Proof (union)

- Define $\delta : Q \times \Sigma \rightarrow Q$ by

  $$\delta((p_1, p_2), a) = (\delta_1(p_1, a), \delta_2(p_2, a)).$$

- It is clear that the language of $M$ is $L_1 \cup L_2$. 
Proof.

Proof (concatenation, star)

- What machine will accept $L_1 \circ L_2$?
- What machine will accept $L_1^*$?
Other Operations

**Definition (Intersection)**

The **intersection** of languages $A$ and $B$ is the language

$$ A \cap B = \{ w \mid w \in A \text{ and } w \in B \}. $$

**Definition (Complement)**

The **complement** of language $A$ is the language

$$ \overline{A} = \{ w \mid w \notin A \}. $$
Theorem (Closure of Regular Languages)

The class of regular languages is closed under the operations of intersection and complementation.
Closure

Proof.

Proof (intersection, complement)

- What machine will accept $L_1 \cap L_2$?
- What machine will accept $\overline{L_1}$?
Read Section 1.1, pages 44 - 47.
Exercises 4, 5, 6, pages 83 - 84.
Problem 34, page 89.

Design a DFA for the language \((A \circ B)^*\), where

\[
A = \{w \mid w \text{ contains an odd number of a's}\}
\]
\[
B = \{w \mid w \text{ contains an odd number of b's}\}
\]