Mapping Reducibility
Lecture 35
Section 5.3

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1. Reducibility

2. Mapping Reducibility

3. The Unrecognizability of $E_{TM}$ and $E_{TM}$

4. Assignment
Outline

1 Reducibility

2 Mapping Reducibility

3 The Unrecognizability of $EQ_{TM}$ and $\overline{EQ}_{TM}$

4 Assignment
We have been “reducing” one problem to another.

Then we claimed that if the reduced problem was decidable, then so was the original problem.

But how do we know that a Turing machine can carry out the reduction?

If a Turing machine cannot perform the reduction, then the original problem may not really be decidable.
Definition (Computable function)

A function $f : \Sigma^* \rightarrow \Sigma^*$ is **computable** if there exists a Turing machine $M$ such that on every input $w$, $M$ halts with $f(w)$, and only $f(w)$, on its tape.
Examples of Computable Functions

Example (Computable functions)

- The following functions are computable.
  - $incr(n) = n + 1$.
  - $decr(n) = \begin{cases} n - 1, & n > 0; \\ 0, & n = 0. \end{cases}$
  - $add(m, n) = m + n$.
  - $sub(m, n) = \begin{cases} m - n, & m \geq n; \\ 0, & m < n. \end{cases}$
  - $mult(m, n) = mn$.
  - $sqrt(n) = \lfloor \sqrt{n} \rfloor$. 
Outline

1. Reducibility
2. Mapping Reducibility
3. The Unrecognizability of $EQ_{TM}$ and $EQ'_{TM}$
4. Assignment
A language $A$ is mapping reducible (or just reducible) to a language $B$ if there exists a computable function $f : \Sigma^* \rightarrow \Sigma^*$ such that for every $w \in \Sigma^*$,

$$w \in A \iff f(w) \in B.$$ 

$f$ is called a reduction of $A$ to $B$. 

This is denoted $A \leq_m B$. 

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Definition

Note that a “yes” answer to the reduced problem (Is $f(w) \in B$?) must correspond to a “yes” answer to the original problem (Is $w \in A$?), and “no” must correspond to “no”.
Example (Reduction)

- Let $\Sigma^* = \{0, 1, \#\}$.
- Let GREATER be the language of binary representations of pairs of integers $(n_1, n_2)$, where $n_1 > n_2$.
- Let POSITIVE be the language of binary representations of positive integers.
- The Turing machine SUBTRACT is a reduction of GREATER to POSITIVE.
- Thus, GREATER $\leq_m$ POSITIVE.
Example of a Reduction

Example (Reduction)

- Let LESS be the language of binary representations of pairs of integers \((n_1, n_2)\), where \(n_1 < n_2\).
- Would SUBTRACT be a reduction of LESS to POSITIVE?
- Is there a way to reduce LESS to GREATER (which reduces to POSITIVE)?
Decidability and Reducibility

**Theorem**

*If B is decidable and $A \leq_m B$, then A is decidable.*

**Proof.**

- Let $D_B$ be a decider for $B$.
- Let $f$ be a reduction of $A$ to $B$.
- Then build a decider $D_A$ for $A$ as follows.
Decidability and Reducibility

\[ D_A \xrightarrow{\langle w \rangle} f \xrightarrow{\langle f(w) \rangle} D_B \]

\[ \text{yes} \quad \text{yes} \quad \text{no} \quad \text{no} \]

\[ f(w) \]

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We showed that $A_{TM} \leq_m E_{TM}$ by building the following Turing machines.

$$M_w \quad x = w \quad \langle M, w \rangle \quad U \quad x \neq w \quad \langle M, w \rangle \quad \text{acc} \quad \text{acc} \quad \text{rej} \quad \text{rej}$$

$$D_A \quad \langle M, w \rangle \quad \text{CONVERT} \quad \langle M_w \rangle \quad D_E \quad \text{yes} \quad \text{no} \quad \text{yes} \quad \text{no}$$
Notice that we needed to reverse the “accept” and “reject” outputs.

That is, if $f : \Sigma^* \rightarrow \Sigma^*$ by $f : \langle M, w \rangle \mapsto \langle M_w \rangle$, then

$$\langle M, w \rangle \in A_{TM} \iff f (\langle M, w \rangle) \notin E_{TM}.$$ 

In other words, we reduced $A_{TM}$ to $\overline{E_{TM}}$, not $E_{TM}$. 
It is ok to build this reversal into the design of a decider. But it is important to note that we did not reduce $A_{TM}$ to $E_{TM}$. If we know that

$$w \in A \iff f(w) \in B.$$ 

we cannot assume that there is a function $g$ such that

$$w \in A \iff g(w) \notin B.$$ 

just because it would be convenient.
Theorem

If \( A \leq_m B \), then

- \( B \) is decidable \( \Rightarrow \) \( A \) is decidable.
- \( A \) is undecidable \( \Rightarrow \) \( B \) is undecidable.
- \( B \) is recognizable \( \Rightarrow \) \( A \) is recognizable.
- \( A \) is unrecognizable \( \Rightarrow \) \( B \) is unrecognizable.
1 Reducibility

2 Mapping Reducibility

3 The Unrecognizability of $EQ_{TM}$ and $EQ_{TM}$

4 Assignment
The Unrecognizability of $EQ_{TM}$ and $\overline{EQ}_{TM}$

**Theorem**

$EQ_{TM}$ and $\overline{EQ}_{TM}$ are both unrecognizable.
The Unrecognizability of $EQ_{TM}$ and $\overline{EQ}_{TM}$

**Proof.**

- Suppose that $EQ_{TM}$ is recognizable.
- Let $R_{EQ}$ be a recognizer of $EQ_{TM}$.
- We will build a recognizer for $\overline{A}_{TM}$. 
Proof.

- Given $M$ and $w$, build the following two machines $M_\emptyset$ and $M_{\langle M, w \rangle}$.
The Unrecognizability of $\mathit{EQ}_{TM}$ and $\overline{\mathit{EQ}}_{TM}$

Proof.

- What is the language of $M_{\varnothing}$?
- What is the language of $M_{\langle M, w \rangle}$?
Proof.

- Now build the Turing machine $M'$:

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\[
\begin{align*}
&M', \langle M, w \rangle \\
&M' \quad \langle M_\varnothing, M_{\langle M, w \rangle} \rangle \\
&\quad \quad R_{EQ} \quad \text{acc} \quad \text{acc}
\end{align*}
\]
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The Unrecognizability of $EQ_{TM}$ and $\overline{EQ}_{TM}$

Proof.

$M$ rejects $w \iff M_\emptyset$ and $M_{\langle M, w \rangle}$ are equivalent.

$\iff R_{EQ}$ accepts $\langle M_\emptyset, M_{\langle M, w \rangle} \rangle$.

$\iff M'$ accepts $M_{\langle M, w \rangle}$.

Thus, $M'$ recognizes $\overline{A_{TM}}$, a contradiction.
To show that $\overline{EQ_{TM}}$ is also unrecognizable, replace $M_{\emptyset}$ with this machine and repeat the argument.
Irreducibility of $A_{TM}$ to $E_{TM}$

**Theorem**

$A_{TM}$ is not mapping reducible to $E_{TM}$.

**Lemma**

If $A \leq_m B$, then $\overline{A} \leq_m \overline{B}$. 
Irreducibility of $A_{TM}$ to $E_{TM}$

**Proof of the lemma.**

- The statement
  \[ w \in A \iff f(w) \in B \]
  is equivalent to the statement
  \[ w \notin A \iff f(w) \notin B \]
  which is equivalent to
  \[ w \in \overline{A} \iff f(w) \in \overline{B}. \]
Irreducibility of $A_{TM}$ to $E_{TM}$

Proof of the theorem.

- Suppose that $A_{TM}$ is mapping reducible to $E_{TM}$.
- Then $\overline{A_{TM}} \leq_m \overline{E_{TM}}$.
- But we know that $\overline{E_{TM}}$ is recognizable and that $\overline{A_{TM}}$ is not recognizable. (How?)
- This is a contradiction. (Why?)
- Therefore, $A_{TM}$ is not mapping reducible to $E_{TM}$. 
Assignment

- Read Section 5.3, pages 234 - 238.
- Exercises 4, 6, page 239.
- Problems 22, 23, 24, 25, page 239.