Other NP-Complete Problems
Lecture 42
Section 7.5

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1. 3SAT is NP-Complete
2. VERTEX COVER
3. MINESWEEPER is NP-Complete
4. Assignment
Outline

1. 3SAT is NP-Complete
2. VERTEX COVER
3. MINESWEEPER is NP-Complete
4. Assignment
Theorem

If A is NP-complete and B is in NP and A is polynomial-time reducible to B, then B is NP-complete.

Proof.

- This follows from the transitivity of $\leq_P$. 
3SAT is NP-Complete

**Theorem**

3SAT is NP-complete.

**Proof.**

- We need only show that SAT is polynomial-time reducible to 3SAT.
- Any boolean expression may be put into 3-conjunctive normal form by the following technique.
3SAT is NP-Complete

Proof.

- Rewrite the each clause

\[ x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_n \]

as

\[ (x_1 \lor x_2 \lor y_1) \land (\overline{y}_1 \lor x_3 \lor y_2) \land \cdots \land (\overline{y}_{n-3} \lor x_{n-1} \lor x_n). \]
3SAT is NP-Complete

Proof.

- If a clause has only 2 literals, then rewrite

\[ x_1 \lor x_2 \]

as

\[ (x_1 \lor x_2 \lor y) \land (x_1 \lor x_2 \lor \overline{y}) \]
3SAT is NP-Complete

Proof.

If a clause has only 1 literal, then rewrite

\[ x_1 \]

as

\[(x_1 \lor y \lor z) \land (x_1 \lor \overline{y} \lor z) \land (x_1 \lor y \lor \overline{z}) \land (x_1 \lor \overline{y} \lor \overline{z}).\]
Proof.

- The expressions $\phi_{\text{cell}}$, $\phi_{\text{start}}$, and $\phi_{\text{accept}}$ are already in conjunctive normal form.
- Apply the above technique to put them in 3-conjunctive normal form.
- The expression $\phi_{\text{move}}$ is not in conjunctive normal form.
Proof.

However, it may be rewritten in conjunctive normal form by repeatedly applying the distributive laws

\[ x \land (y \lor z) = (x \land y) \lor (x \land z) \]

and

\[ x \lor (y \land z) = (x \lor y) \land (x \lor z). \]
3SAT is NP-Complete

Proof.

For example,

\[(x \land y) \lor (z \land w) = [(x \land y) \lor z] \land [(x \land y) \lor w] \]
\[= [(x \lor z) \land (y \lor z)] \land [(x \lor w) \land (y \lor w)] \]
\[= (x \lor z) \land (y \lor z) \land (x \lor w) \land (y \lor w).\]
Proof.

Therefore, the expression

$$\phi_{\text{cell}} \land \phi_{\text{start}} \land \phi_{\text{accept}} \land \phi_{\text{move}}$$

can be rewritten in 3-conjunctive normal form.

Therefore, 3SAT is NP-complete.
We have shown that SAT is NP-complete which means that

\[ A \leq_P SAT \]

for every NP problem A.

We have also shown that

\[ SAT \leq_P 3SAT \leq_P CLIQUE \]

which means that 3SAT and CLIQUE are also NP-complete.
Now we will show that

\[ \text{CLIQUE} \leq_p \text{VERTEX COVER}, \]

which will make \text{VERTEX COVER} NP-complete.
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Definition (Vertex Cover)

Let $G$ be a graph. A set of vertices $C$ is a vertex cover of $G$ if every edge of $G$ is incident to some vertex in $C$.

The VERTEX COVER Problem

Given a graph $G$ and an integer $k$, does $G$ have a vertex cover of $k$ vertices?
Given a graph $G$ of $n$ vertices and an integer $k$, we reduce VERTEX COVER to CLIQUE.

- Let $\overline{G}$ be the complementary graph.
- That is, $e$ is an edge of $\overline{G}$ if and only if $e$ is not an edge of $G$.
- Then solve CLIQUE for $\overline{G}$ and the integer $n - k$. 
Find a vertex cover of $G$ of size 4
Consider the complementary graph $\bar{G}$
Consider the complementary graph $\overline{G}$
VERTEX COVER $\leq_p$ CLIQUE

Find a clique of $\overline{G}$ of size $8 - 4 = 4$
The complementary vertices the clique...
… form a vertex cover of $G$ of size 4
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MINESWEEPER is NP-Complete

- The Minesweeper Consistency Problem is NP-complete.
- We can prove that by reducing 3SAT to MINESWEEPER.
- See MINESWEEPER and NP-completeness
- See The Clay Mathematics Institute
There are many other NP-complete problems.

See NP-Complete

See List of NP-Complete Problems
1 3SAT is NP-Complete

2 VERTEX COVER

3 MINESWEEPER is NP-Complete

4 Assignment
Assignment

- Read Section 7.4, pages 276 - 283.
- Problems 26, 27, 29, 30, 31, 36, 37, 38, page 297 - 298.