Outline

1. The Regular Operations
2. Closure Properties
3. Assignment
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The Regular Operations

Definition (Union of languages)
The union of languages \( A \) and \( B \) is the language

\[
A \cup B = \{ w \mid w \in A \text{ or } w \in B \}.
\]

Definition (Concatenation of languages)
The concatenation of languages \( A \) and \( B \) is the language

\[
A \circ B = \{ uv \mid u \in A \text{ and } v \in B \}.
\]

Definition (Kleene star of a language)
The Kleene star of a language \( A \) is the language

\[
A^* = \{ w_1 w_2 \ldots w_k \mid w_i \in A \text{ and } k \geq 0 \}.
\]
The Regular Operations

- We often abbreviate $A \circ B$ as $AB$.
- Then we may abbreviate $AA$ as $A^2$, $AAA$ as $A^3$, and so on.
- The Kleene star of $A$ can be written as

$$A^* = \{\varepsilon\} \cup A \cup A^2 \cup A^3 \cup \cdots.$$
Examples

Example (Regular operations)

- Let \( A = \{ w \mid w \text{ contains an even number of } a\text{’s} \} \).
- Let \( B = \{ w \mid w \text{ contains an even number of } b\text{’s} \} \).
- Describe the languages
  - \( A \cup B \)
  - \( A \circ B \)
  - \( A^* \)
  - \( (A \cup B)^* \)
  - \( (A \circ B)^* \)
  - \( (A^*)^* \)
Example (Regular operations)

Design finite automata that accept

- $A \cup B$
- $A \circ B$
- $A^*$
- $(A \cup B)^*$
- $(A \circ B)^*$
- $(A^*)^*$
Example (Regular operations)

A DFA for $A \cup B$. 

![DFA Diagram]

- States $A$ and $B$ are represented by circles.
- Transitions are labeled with 'a' and 'b'.
- The DFA accepts strings that are in $A$ or $B$ or both.
Examples

- Design a DFA for $A \cap B$. 
Example (Regular operations)

A DFA for $A \circ B$. 

\begin{center}
\begin{tikzpicture}
  \node[state, initial] (q0) at (0,0) {$a, b$};
  \node[state] (q1) at (2,0) {$a$};
  \node[state] (q2) at (2,2) {$b$};
  \node[state] (q3) at (0,2) {$a$};
  \node[state] (q4) at (0,4) {$a$};

  \draw[->] (q0) edge (q1)
              (q1) edge (q2)
              (q2) edge (q3)
              (q3) edge (q0)
              (q1) edge (q2)
              (q2) edge (q3);
\end{tikzpicture}
\end{center}
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Theorem (Closure of Regular Languages)

*The class of regular languages is closed under the operations of union, concatenation, and star.*
Proof.

Proof (union)

Let $M_1 = \{Q_1, \Sigma_1, \delta_1, q_1, F_1\}$ be a DFA whose language is $L_1$.
Let $M_2 = \{Q_2, \Sigma_2, \delta_2, q_2, F_2\}$ be a DFA whose language is $L_2$.
We will define a DFA $M$ whose language is $L_1 \cup L_2$.
Let $M = \{Q, \Sigma, \delta, q_0, F\}$ where
- $Q = Q_1 \times Q_2$.
- $\Sigma = \Sigma_1 \cup \Sigma_2$.
- $q_0 = (q_1, q_2)$.
- $F = \{(p_1, p_2) \mid p_1 \in F_1 \text{ or } p_2 \in F_2\}$. 
Proof.

Proof (union)

- Define $\delta : Q \times \Sigma \rightarrow Q$ by
  $$\delta((p_1, p_2), a) = (\delta_1(p_1, a), \delta_2(p_2, a)).$$

- It is clear that the language of $M$ is $L_1 \cup L_2$. 
Proof.

Proof (concatenation, star)
- What machine will accept $L_1 \circ L_2$?
- What machine will accept $L_1^*$?
Example (Concatenation Example)

- Let
  \[ L_1 = \{ w \in \Sigma^* \mid w \text{ has an even number of a's} \} \]
  and
  \[ L_2 = \{ w \in \Sigma^* \mid w \text{ has an even number of b's} \} \]
- How would a DFA for \( L_1 L_2 \) process the strings \textbf{ababb} and \textbf{ababbbb}?
Other Operations

Definition (Intersection)
The intersection of languages $A$ and $B$ is the language

$$A \cap B = \{ w \mid w \in A \text{ and } w \in B \}.$$ 

Definition (Complement)
The complement of language $A$ is the language

$$\overline{A} = \{ w \in \Sigma^* \mid w \notin A \}.$$
Theorem (Closure of Regular Languages)

The class of regular languages is closed under the operations of intersection and complementation.
Proof.

Proof (intersection, complement)

- What machine will accept $L_1 \cap L_2$?
- What machine will accept $\overline{L}_1$?
1. The Regular Operations
2. Closure Properties
3. Assignment
Design a DFA for the language \((A \circ B)^*\), where

\[
A = \{ w \mid w \text{ contains an odd number of } a\text{'s} \} \\
B = \{ w \mid w \text{ contains an odd number of } b\text{'s} \}
\]