Outline

1. Assignment
2. The Regular Operations
3. Closure Properties
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Assignment

Homework

- Read Section 1.1, pages 44 - 47.
- Exercises 4, 5, 6, pages 83 - 84.
- Problem 34, page 89.
- Design a DFA for the language \((A \circ B)^*\), where

\[
A = \{ w \mid w \text{ contains an odd number of a's} \}
\]

\[
B = \{ w \mid w \text{ contains an odd number of b's} \}
\]
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The Regular Operations

Definition (Union of languages)
The union of languages $A$ and $B$ is the language

$$A \cup B = \{ w \mid w \in A \text{ or } w \in B \}.$$  

Definition (Concatenation of languages)
The concatenation of languages $A$ and $B$ is the language

$$A \circ B = \{ uv \mid u \in A \text{ and } v \in B \}.$$  

Definition (Kleene star of a language)
The Kleene star of a language $A$ is the language

$$A^* = \{ w_1 w_2 \ldots w_k \mid w_i \in A \text{ and } k \geq 0 \}.$$
We often abbreviate $A \circ B$ as $AB$.
Then we may abbreviate $AA$ as $A^2$, $AAA$ as $A^3$, and so on.
The Kleene star of $A$ can be written as

$$A^* = \{\varepsilon\} \cup A \cup A^2 \cup A^3 \cup \cdots.$$
Example (Regular operations)

Let

\[ A = \{ w \mid w \text{ contains exactly one } a \text{ and any number of } b \text{’s} \} \]
\[ B = \{ w \mid w \text{ contains exactly one } b \text{ and any number of } a \text{’s} \} . \]

Describe the languages

- \( A \cup B \)
- \( A \circ B \)
- \( A^* \)
- \( (A \cup B)^* \)
- \( (A \circ B)^* \)
- \( A^* \circ B^* \)
Example (Regular operations)

- Let
  
  \[ A = \{ w \mid w \text{ contains exactly one } a \text{ and any number of } b's \} \]
  
  \[ B = \{ w \mid w \text{ contains exactly one } b \text{ and any number of } a's \}. \]

- Design a DFA for \( A \cup B \).
- Design a DFA for \( A \circ B \).
- Design a DFA for \( A^* \).
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Theorem (Closure of Regular Languages)

The class of regular languages is closed under the operations of union, concatenation, and star.
Proof.

Proof (union)

Let $M_1 = \{Q_1, \Sigma_1, \delta_1, q_1, F_1\}$ be a DFA whose language is $L_1$.
Let $M_2 = \{Q_2, \Sigma_2, \delta_2, q_2, F_2\}$ be a DFA whose language is $L_2$.
We will define a DFA $M$ whose language is $L_1 \cup L_2$.
Let $M = \{Q, \Sigma, \delta, q_0, F\}$ where
- $Q = Q_1 \times Q_2$.
- $\Sigma = \Sigma_1 \cup \Sigma_2$.
- $q_0 = (q_1, q_2)$.
- $F = \{(p_1, p_2) \mid p_1 \in F_1 \text{ or } p_2 \in F_2\}$. 
Proof.

Proof (union)

- Define $\delta : Q \times \Sigma \rightarrow Q$ by

  $$\delta((p_1, p_2), a) = (\delta_1(p_1, a), \delta_2(p_2, a)).$$

- It is clear that the language of $M$ is $L_1 \cup L_2$. 

Robb T. Koether  (Hampden-Sydney College)  Finite Automata - Regular Operations  Fri, Sep 5, 2014  13 / 19
Closure

Proof.

Proof (concatenation, star)

- What machine will accept $L_1 \circ L_2$?
- What machine will accept $L_1^*$?
Example (Concatenation Example)

Let

\[ L_1 = \{ w \in \Sigma^* \mid w \text{ has an even number of } a's \} \]

and

\[ L_2 = \{ w \in \Sigma^* \mid w \text{ has an even number of } b's \} \]

How would a DFA for \( L_1 L_2 \) process the strings \texttt{ababb} and \texttt{ababbb}?
**Definition (Intersection)**

The intersection of languages $A$ and $B$ is the language

$$A \cap B = \{ w \mid w \in A \text{ and } w \in B \}.$$ 

**Definition (Complement)**

The complement of language $A$ is the language

$$\overline{A} = \{ w \in \Sigma^* \mid w \notin A \}.$$
Examples

Let

\[ A = \{ w \mid w \text{ contains exactly one } a \text{ and any number of } b\text{'s} \} \]
\[ B = \{ w \mid w \text{ contains exactly one } b \text{ and any number of } a\text{'s} \}. \]

Design a DFA for \( A^* \cap B^* \).
Theorem (Closure of Regular Languages)

*The class of regular languages is closed under the operations of intersection and complementation.*
Proof.

Proof (intersection, complement)

- What machine will accept $L_1 \cap L_2$?
- What machine will accept $\overline{L_1}$?