Mathematical Preliminaries and Notation

Lecture 2
Section 1.1

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1. Functions and Relations
2. Equivalence Relations
3. Graphs
4. Order of a Function
5. Mathematical Induction
6. Assignment
Definition (Function)

A function from a set $A$ to a set $B$ is a subset of $A \times B$ with the property that for every $x \in A$, there is exactly one $y \in B$ such that $(x, y)$ is in the function. We denote this as $f : A \rightarrow B$ and $f(x) = y$.

Definition (Domain)

The domain of $f : A \rightarrow B$ is the set $A$.

Definition (Codomain)

The codomain of $f : A \rightarrow B$ is the set $B$. 
Functions and Relations

Definition (One-to-one)
A function $f$ is one-to-one if

$$f(x) = f(y) \Rightarrow x = y.$$  

Equivalently,

$$x \neq y \Rightarrow f(x) \neq f(y).$$

Definition (Onto)
A function $f : A \rightarrow B$ is onto if for every $y \in B$, there is $x \in A$ such that $f(x) = y$. 
Definition (One-to-one correspondence)

A function \( f : A \to B \) is one-to-one correspondence if it is one-to-one and onto.
Equivalence Relations

Definition (Binary relation)
A binary relation on a set $A$ is a subset of $A \times A$.

Definition (Equivalence relation)
An equivalence relation on a set $A$ is a binary relation $R$ on $A$ with the following properties:

- Reflexivity: $(x, x) \in R$ for all $x \in A$.
- Symmetry: $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in A$.
- Transitivity: $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$ for all $x, y, z \in A$. 
Equivalence Relations

Definition (Equivalence class)
Given an equivalence relation $R$ on a set $A$ and an element $x \in A$, the equivalence class of $x$ is the set

$$[x] = \{ y \in A \mid (x, y) \in R \}.$$
We may define two computer programs to be equivalent if they produce the same output whenever the inputs are the same.

Show that this is an equivalence relation.

Describe the equivalence class of the “Hello, world” program under this relation.
Outline

1. Functions and Relations
2. Equivalence Relations
3. Graphs
4. Order of a Function
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Graphs

Definition (Graph)
A graph is a pair \((V, E)\) where \(V\) is a set of vertices and \(E\) is a set of edges.

Definition (Directed graph)
A directed graph is a graph in which each edge has a direction from one vertex to another.
A graph may be used to represent a relation.
Or a function, because a function is a relation.
Draw a vertex for every element in $A$.
If $a$ has the relation to $b$, then draw an edge from $a$ to $b$. 
An Interesting Example

Let $\mathbb{Z}_3$ represent the integers modulo 3: \{0, 1, 2\} and let $f: \mathbb{Z}_3 \times \mathbb{Z}_3 \rightarrow \mathbb{Z}_3$ be multiplication modulo 3.

We could list all possible products:

\[
\begin{align*}
    f(0, 0) &= 0 & f(1, 0) &= 0 & f(2, 0) &= 0 \\
    f(0, 1) &= 0 & f(1, 1) &= 1 & f(2, 1) &= 2 \\
    f(0, 2) &= 0 & f(1, 2) &= 2 & f(2, 2) &= 1
\end{align*}
\]
An Interesting Example

Or we could make a table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
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</thead>
<tbody>
<tr>
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<td>1</td>
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<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
An Interesting Example

- Or we could draw a graph:
An Interesting Example

- Or we could draw a better graph:
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Definition (Order at most, Big Oh)

Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ and $g: \mathbb{Z} \rightarrow \mathbb{Z}$. Then $f$ has order at most $g$ if, for some positive constant $c$ and for $n$ sufficiently large,

$$f(n) \leq c|g(n)|.$$  

This is denoted by $f(n) = O(g(n))$. 
Definition (Order at least, Big Omega)

Let \( f : \mathbb{Z} \to \mathbb{Z} \) and \( g : \mathbb{Z} \to \mathbb{Z} \). Then \( f \) has order at least \( g \) if, for some positive constant \( c \) and for \( n \) sufficiently large,

\[
f(n) \geq c|g(n)|.
\]

This is denoted by \( f(n) = \Omega(g(n)) \).
Definition (Same order, Big Theta)

Let \( f : \mathbb{Z} \to \mathbb{Z} \) and \( g : \mathbb{Z} \to \mathbb{Z} \). Then \( f \) has the same order as \( g \) if, for some positive constants \( c_1 \) and \( c_2 \) and for \( n \) sufficiently large,

\[
c_1 |g(n)| \leq f(n) \leq c_2 |g(n)|.
\]

- This is denoted by \( f(n) = \Theta(g(n)) \).
Example

Let $f(n) = 3n^2 + n$ and $g(n) = 5n^3 + 2n^2 + 4$.

Then

\[
\begin{align*}
  f(n) &= O(g(n)), \\
  g(n) &= \Omega(f(n)), \\
  f(n) &= \Theta(n^2), \\
  g(n) &= \Theta(n^3).
\end{align*}
\]
Given a proposition $P(n)$ about a positive integer $n$, it may be possible to prove $P(n)$ to be true for all positive integers by using mathematical induction.

- Show that $P(1)$ is true.
- Show that if $P(k)$ is true for some $k \geq 1$, then $P(k + 1)$ must also be true.
- It would follow that $P(n)$ is true for all positive integers $n$. 
Examples

Example

- Prove that $2^n > n$ for all $n \geq 1$.
- Prove that $2^n > n^2$ for all $n \geq 7$.
- Prove that $2^n > n^3$ for all $n \geq 10$. 
Assignment

- Read Section 1.1.
- Section 1.1 Exercises: 22, 24, 31, 36, 41, 43.