Annuities

Lecture 38

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Outline

Definitions

Definition (Annuity (Stolen from Investopedia))

"An annuity is a financial product sold by financial institutions that is designed to accept and grow funds from an individual and then, upon annuitization, pay out a stream of payments to the individual at a later point in time."

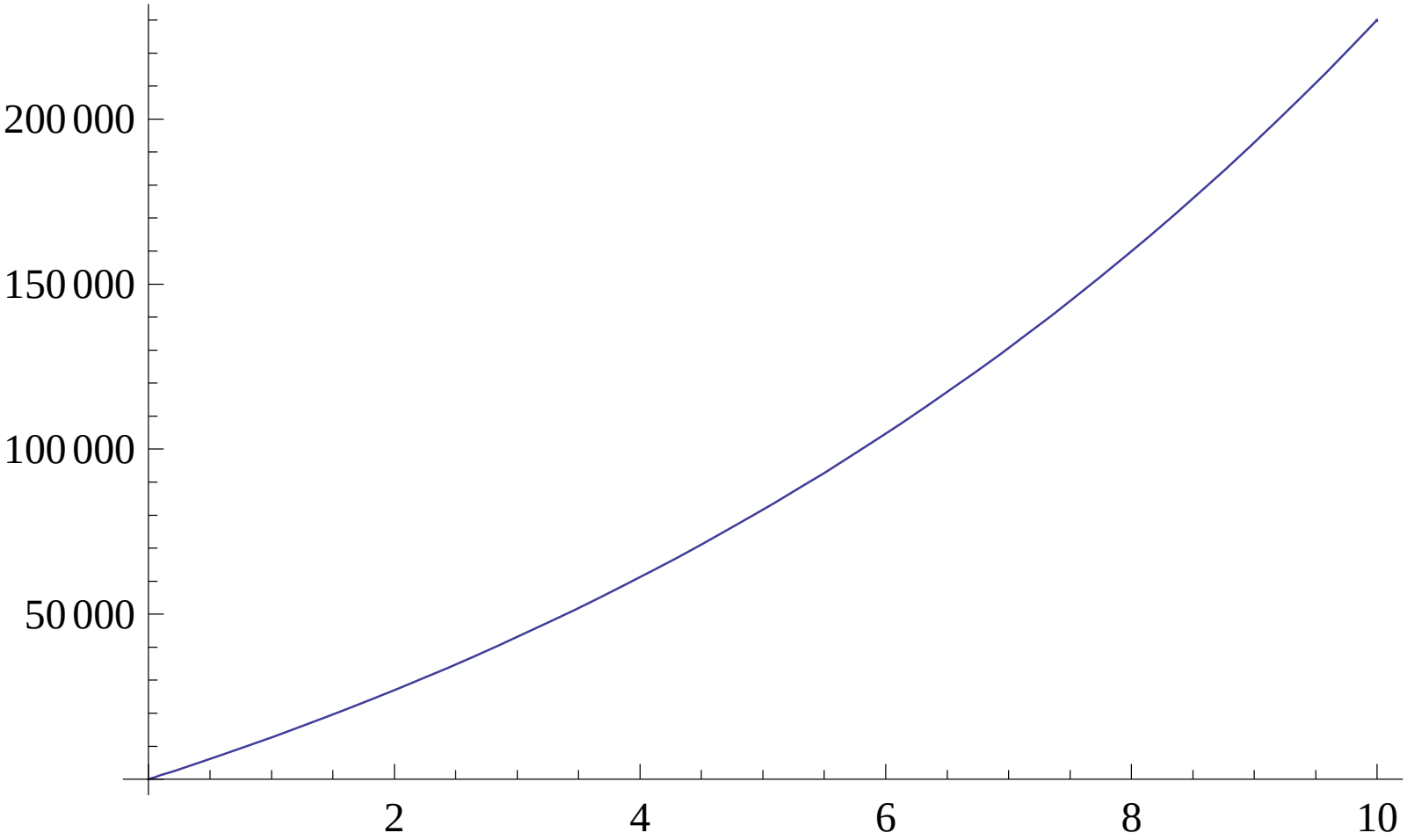
- Typically, a retirement plan is an annuity – You invest over your working life and then withdraw from it during retirement.
- One could establish an annuity to pay for a child's college education – You invest for 18 years and withdraw over the following 4 years.

Annuities

- An annuity has two stages.
  - The investment stage.
  - The withdrawal stage.
- During the investment stage, the balance grows.
- During the withdrawal stage, the balance diminishes.

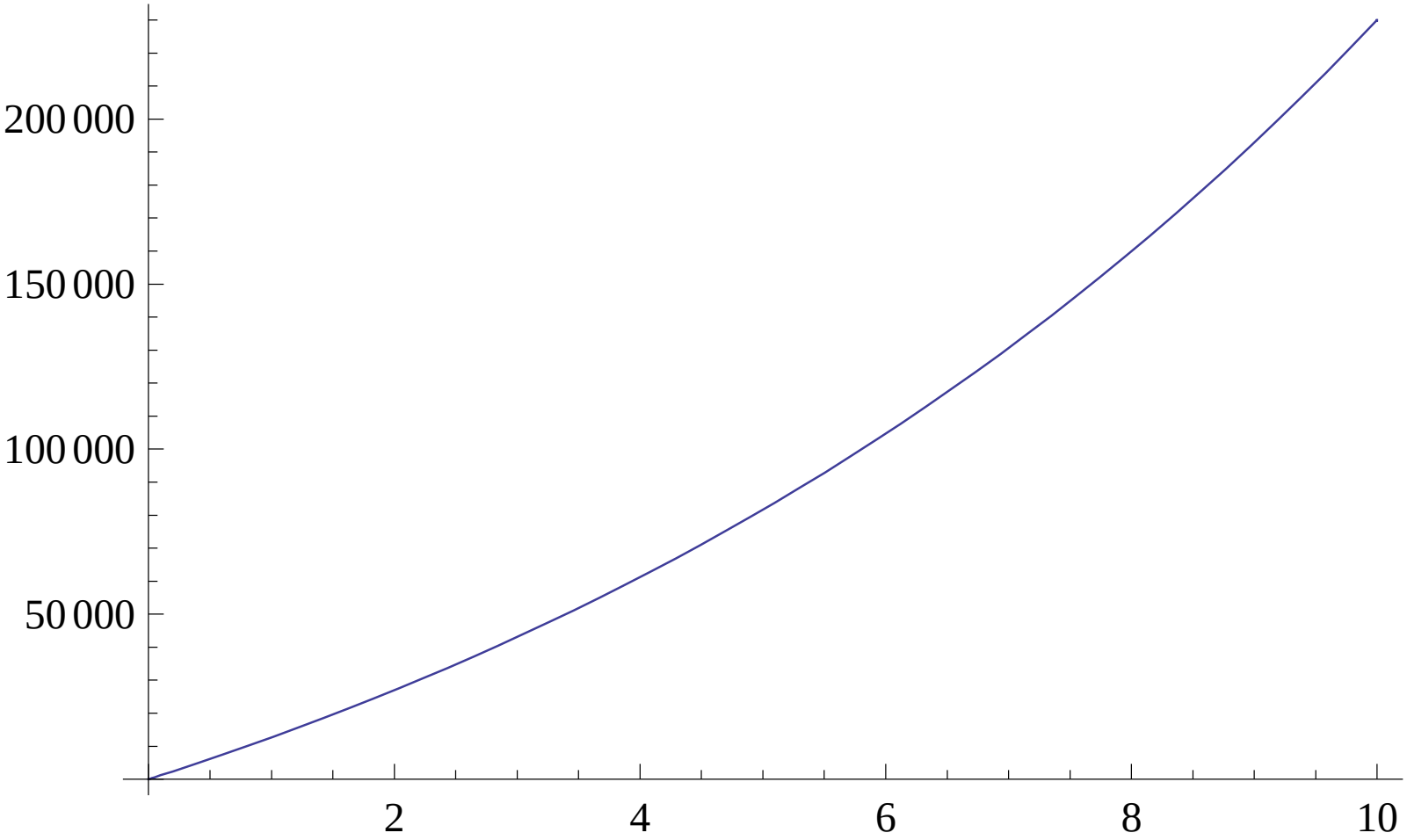
Outline

The Effect of Time



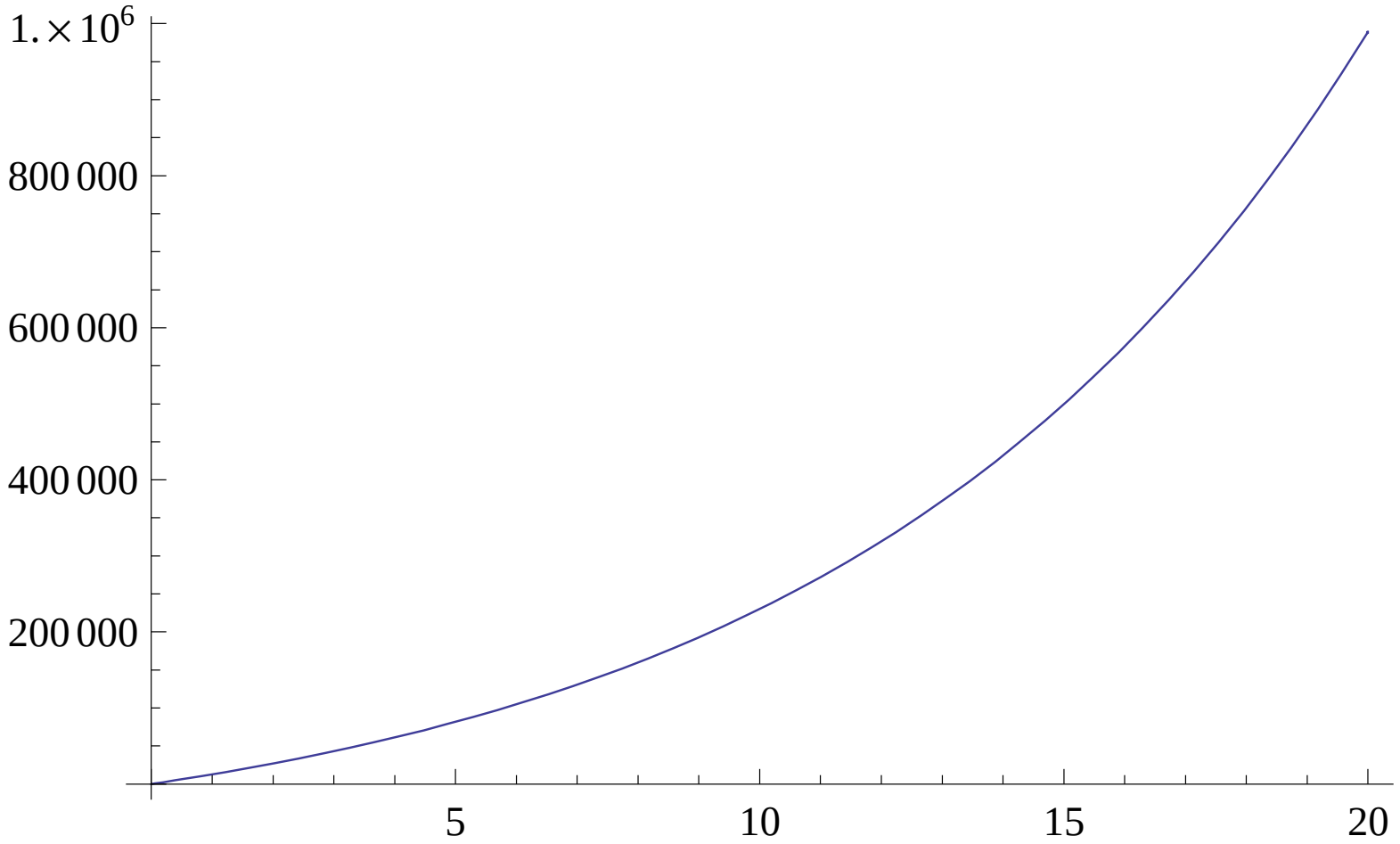
Invest $1,000 per month at 12% for 10 years

The Effect of Time



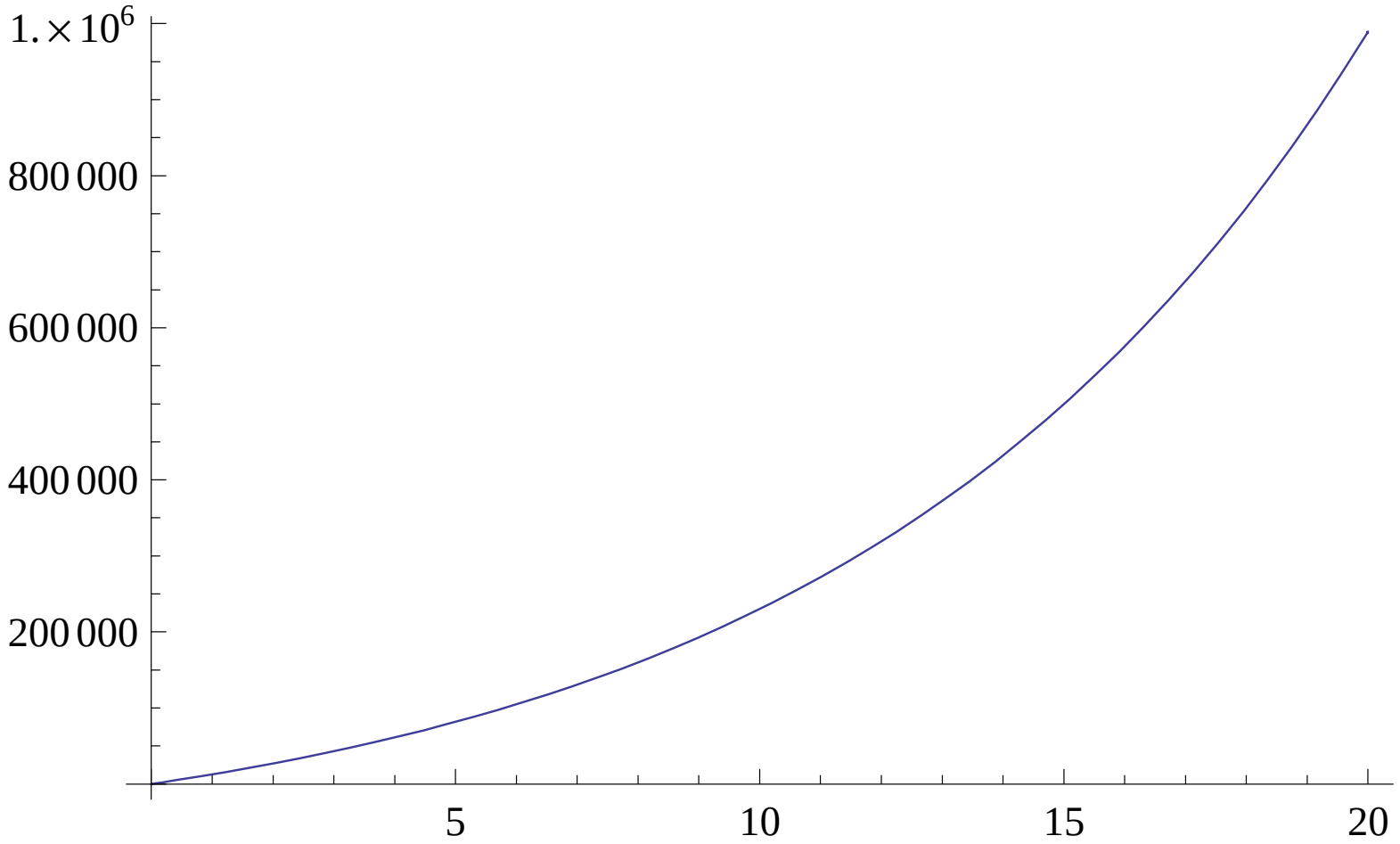
Withdraw $2,532.92 per month for 20 years

The Effect of Time



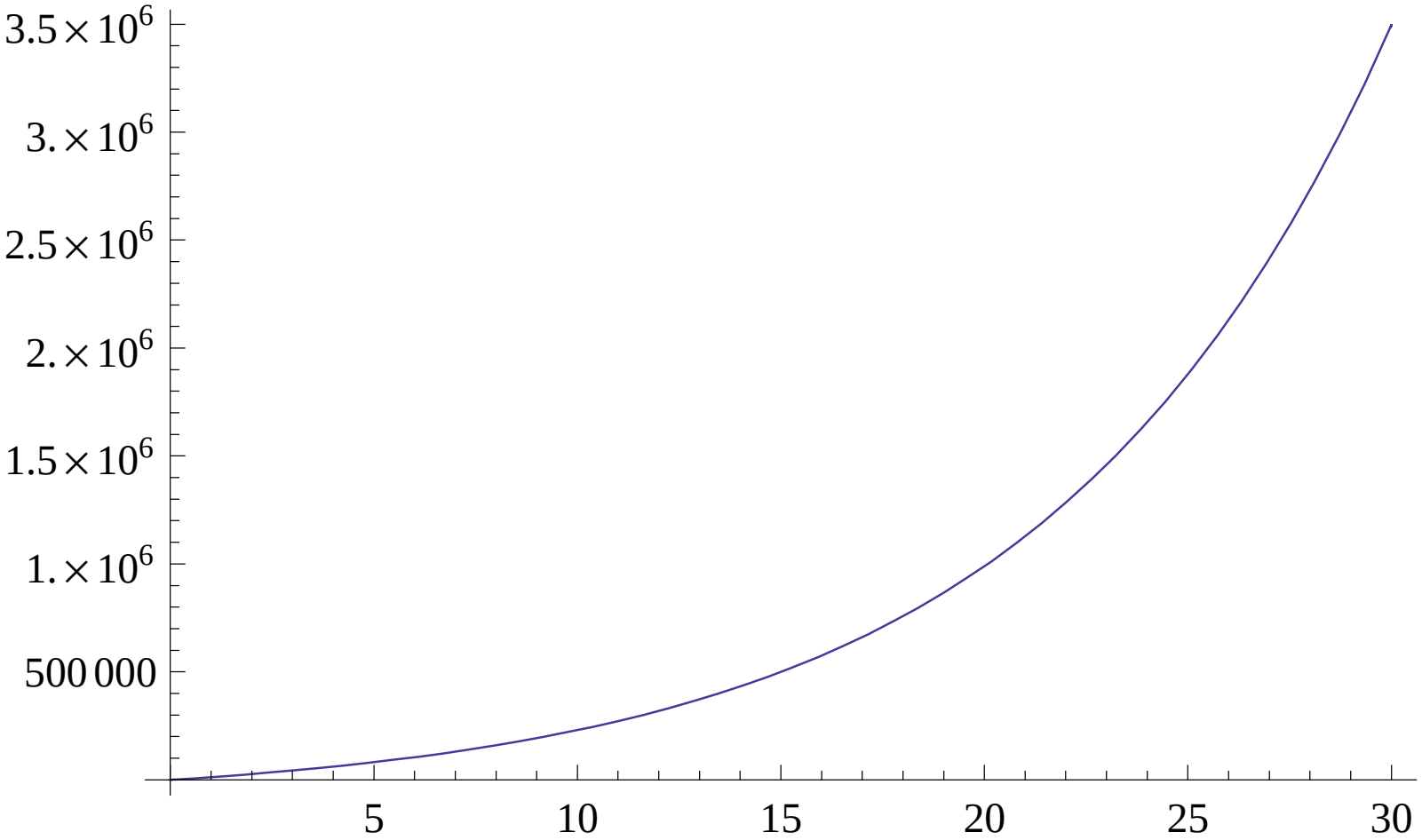
Invest $1,000 per month at 12% for 20 years

The Effect of Time



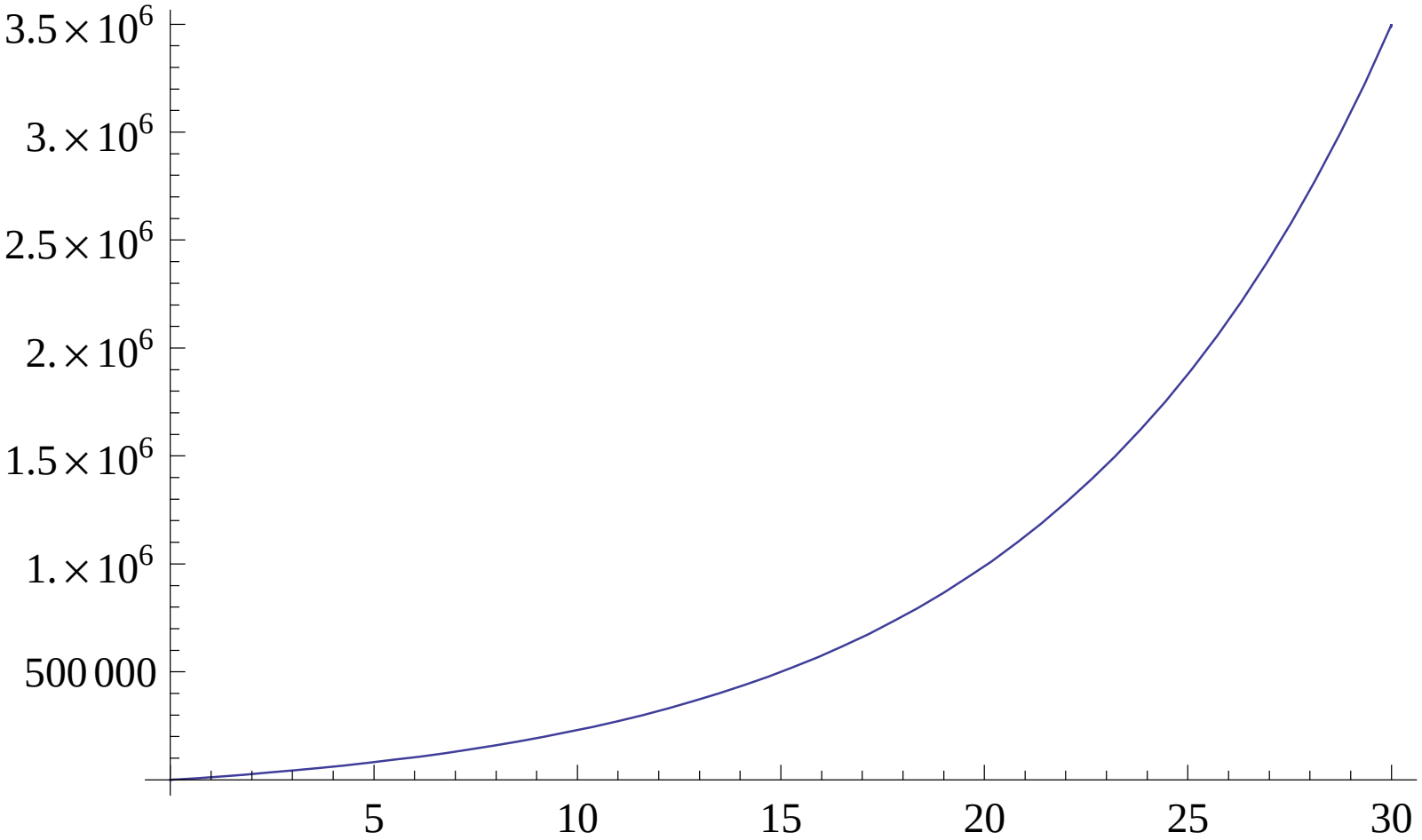
Withdraw $10,892.60 per month for 20 years

The Effect of Time



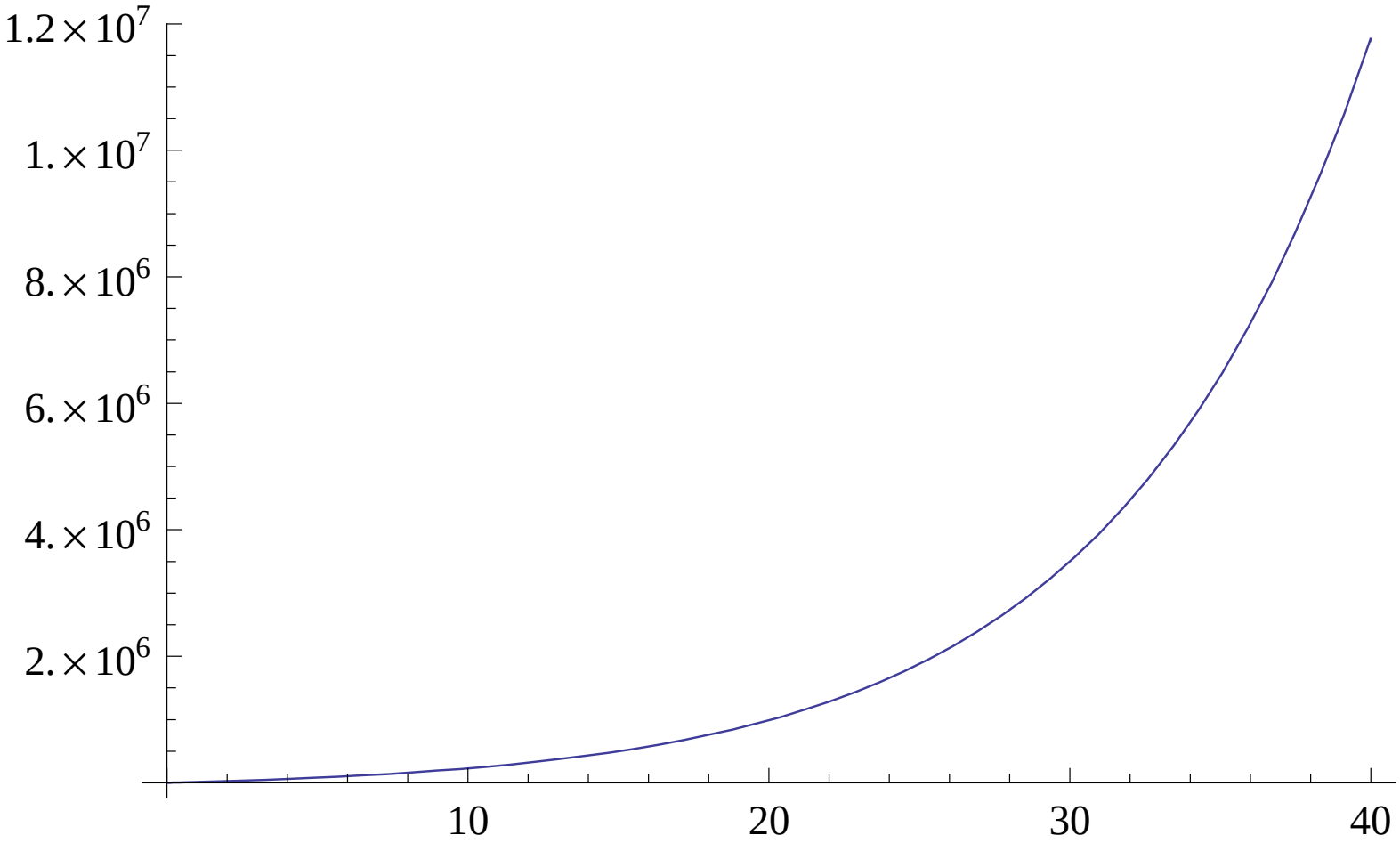
Invest $1,000 per month at 12% for 30 years

The Effect of Time



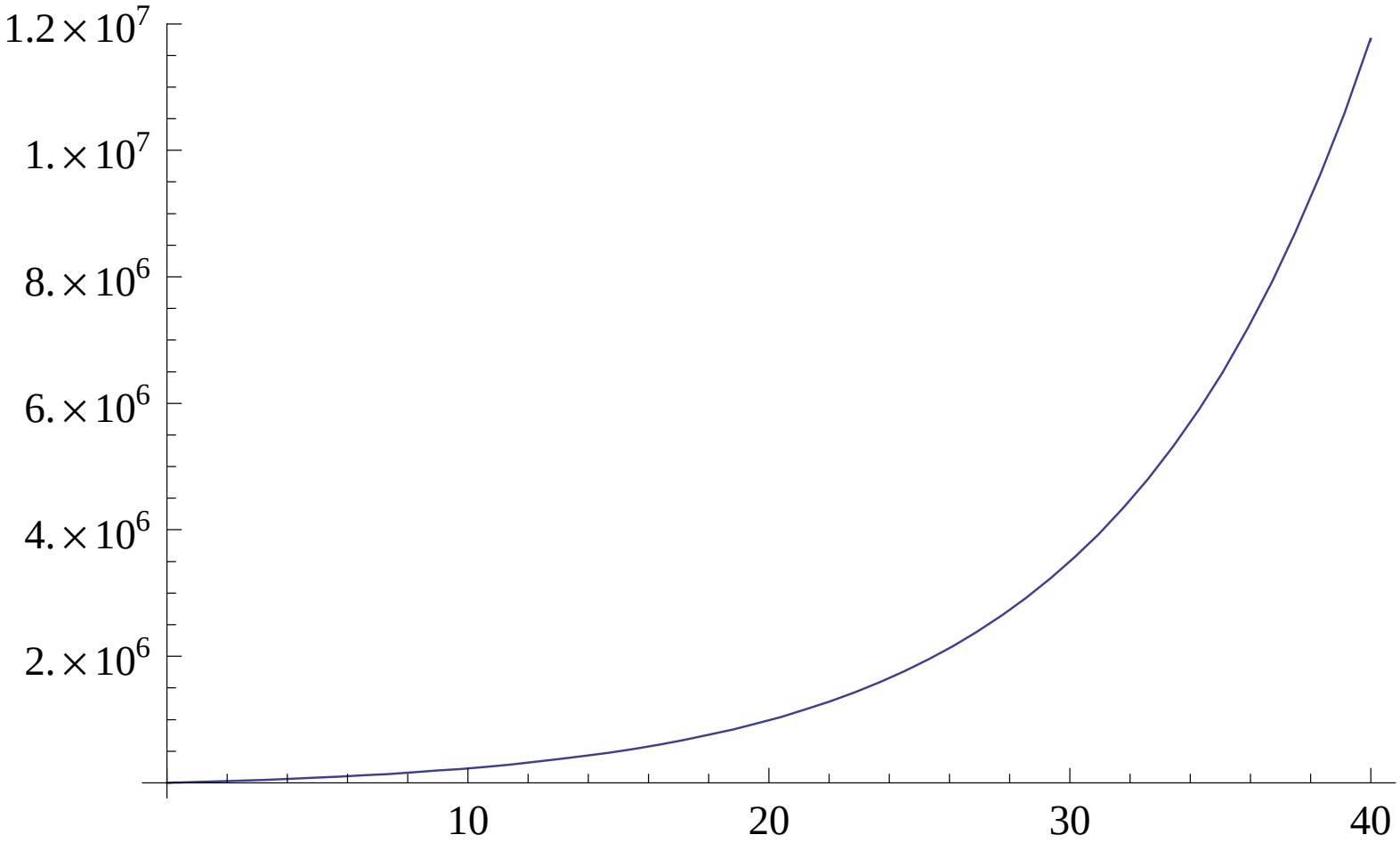
Withdraw $38,482.60 per month for 20 years

The Effect of Time



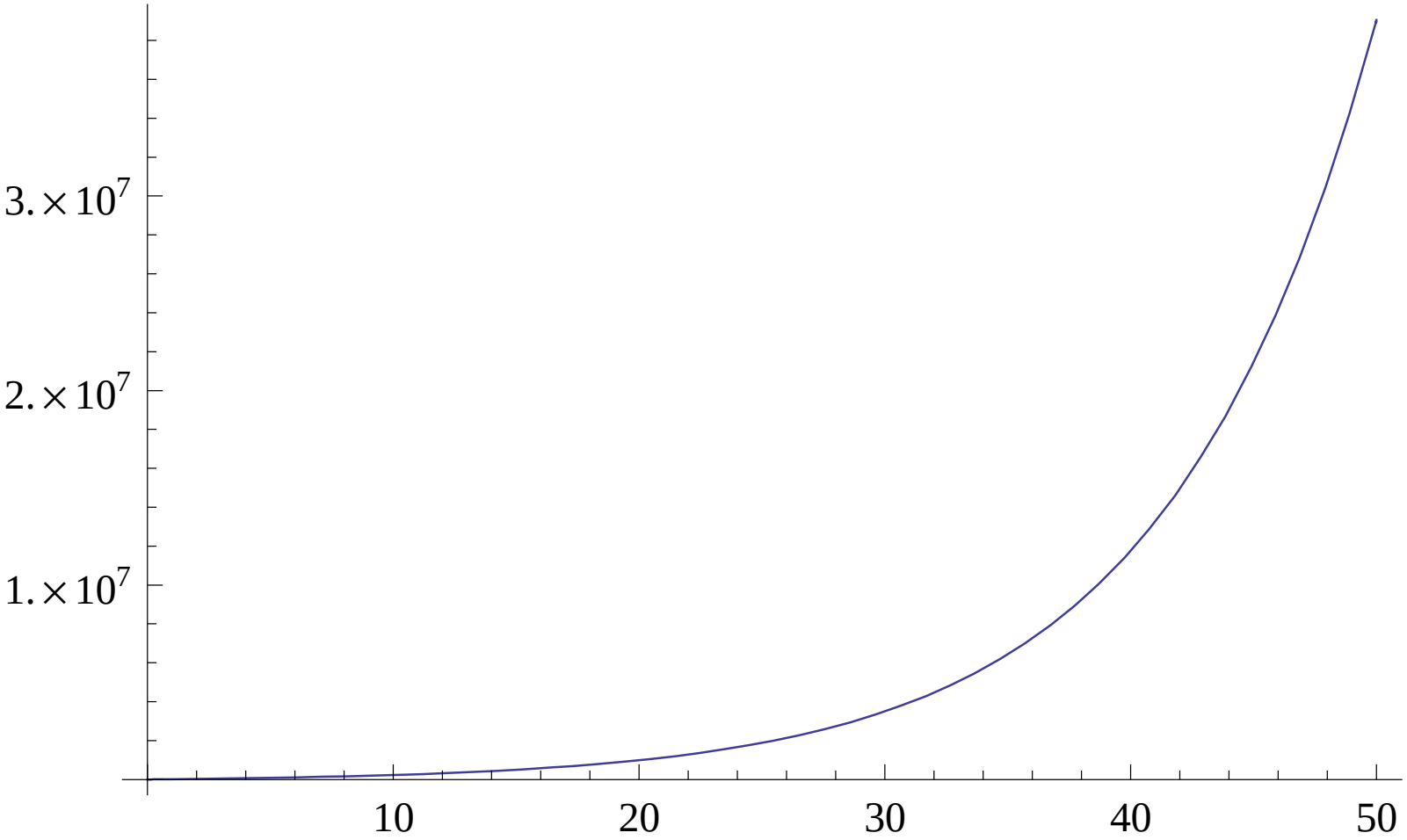
Invest $1,000 per month at 12% for 40 years

The Effect of Time



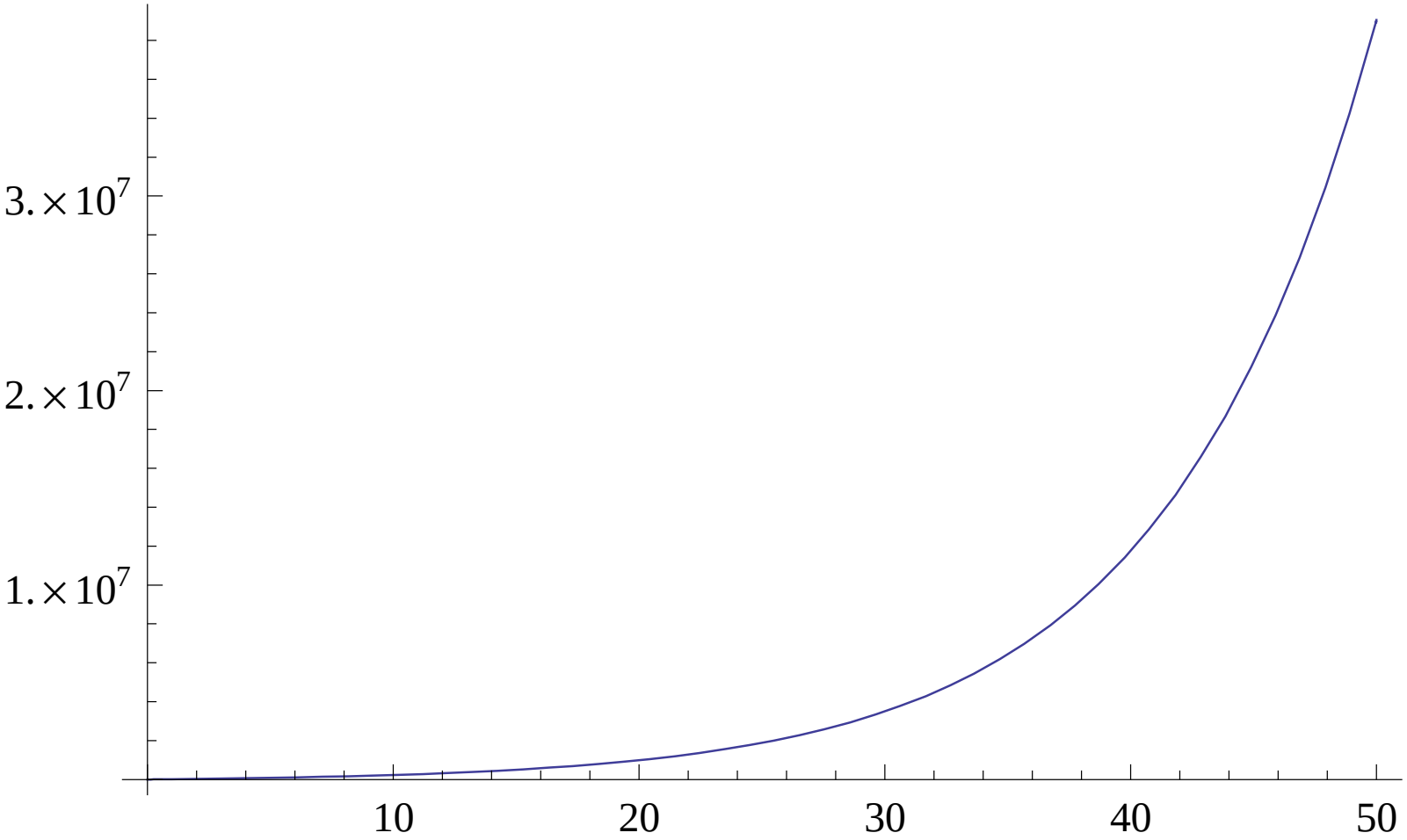
Withdraw $129,540 per month for 20 years

The Effect of Time



Invest $1,000 per month at 12% for 50 years

The Effect of Time



Withdraw $430,066 per month for 20 years

Outline

First Annuity Formula

- The formula for the future value while investing:

$$F = \frac{P((1 + r^*)^n - 1)}{r^*},$$

where F is the future value of the annuity, P is the amount investment per period, r^* is the interest rate *per period*, and n is the number of deposits (one per period).

Example

Example (Three Deposits)

- Let the annual interest rate be 10%.
- Invest $1000 each year for 3 years.

Example

Example (Three Deposits)

The future value is

$$F = \frac{P((1 + r^*)^n - 1)}{r^*}$$

Example

Example (Three Deposits)

The future value is

$$F = \frac{P((1+r^*)^n - 1)}{r^*}$$
$$= \frac{1000((1.10)^3 - 1)}{0.10}$$

Example

Example (Three Deposits)

The future value is

$$\begin{aligned} F &= \frac{P((1+r^*)^n - 1)}{r^*} \\ &= \frac{1000((1.10)^3 - 1)}{0.10} \\ &= 3310.00. \end{aligned}$$

Example

Example (Three Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	$0	$0	$1000	$1000

Example

Example (Three Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	$0	$0	$1000	$1000
2	$1000	$100	$1000	$2100

Example

Example (Three Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	$0	$0	$1000	$1000
2	$1000	$100	$1000	$2100
3	$2100	$210	$1000	$3310

Outline

Second Annuity Formula

- The formula for the amount to withdraw each period:

$$M = \frac{Pr^*}{1 - (1 + r^*)^{-n}},$$

where M is the amount withdrawn per period, P is the amount in the annuity when the withdrawals begin, r is the rate per period, and n is the number of withdrawals (one per period).

Example

Example (Three Withdrawals)

- Continuing the example, the person has accumulated $3310 after 3 years.
- How much can he withdraw each year for 3 years?

Example

Example (Three Withdrawals)

The amount withdrawn is

$$M = \frac{Pr^*}{1 - (1 + r^*)^{-n}}$$

Example

Example (Three Withdrawals)

The amount withdrawn is

$$M = \frac{Pr^*}{1-(1+r^*)^{-n}}$$
$$= \frac{(3310)(.10)}{1-(1.10)^{-3}}$$

Example

Example (Three Withdrawals)

The amount withdrawn is

$$\begin{aligned} M &= \frac{Pr^*}{1-(1+r^*)^{-n}} \\ &= \frac{(3310)(.10)}{1-(1.10)^{-3}} \\ &= 1331.00. \end{aligned}$$

Example

Example (Three Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	$3310	$331	$1331	$2310

Example

Example (Three Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	$3310	$331	$1331	$2310
2	$2310	$231	$1331	$1210

Example

Example (Three Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	$3310	$331	$1331	$2310
2	$2310	$231	$1331	$1210
3	$1210	$121	$1331	$0

Outline

Example

Example (10-Year Example)

- Suppose we invest $1000.00 each year at 12% for 10 years.
- Then we withdraw from the account a fixed amount (to be determined) for the next 10 years.
- Trace through the calculations.

Example

Example (Three Deposits)

The future value is of the annuity is

$$F = \frac{P((1 + r^*)^n - 1)}{r^*}$$

Example

Example (Three Deposits)

The future value is of the annuity is

$$\begin{aligned} F &= \frac{P((1+r^*)^n - 1)}{r^*} \\ &= \frac{1000((1.12)^{10} - 1)}{0.12} \end{aligned}$$

Example

Example (Three Deposits)

The future value is of the annuity is

$$\begin{aligned} F &= \frac{P((1+r^*)^n - 1)}{r^*} \\ &= \frac{1000((1.12)^{10} - 1)}{0.12} \\ &= 17,548.74 \end{aligned}$$

Example

Example (Ten Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	0.00	0.00	1000.00	1000.00

Example

Example (Ten Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	0.00	0.00	1000.00	1000.00
2	1000.00	120.00	1000.00	2120.00

Example

Example (Ten Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	0.00	0.00	1000.00	1000.00
2	1000.00	120.00	1000.00	2120.00
3	2120.00	254.40	1000.00	3374.40

Example

Example (Ten Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	0.00	0.00	1000.00	1000.00
2	1000.00	120.00	1000.00	2120.00
3	2120.00	254.40	1000.00	3374.40
4	3374.40	404.93	1000.00	4779.33

Example

Example (Ten Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	0.00	0.00	1000.00	1000.00
2	1000.00	120.00	1000.00	2120.00
3	2120.00	254.40	1000.00	3374.40
4	3374.40	404.93	1000.00	4779.33
5	4779.33	573.52	1000.00	6352.85

Example

Example (Ten Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	0.00	0.00	1000.00	1000.00
2	1000.00	120.00	1000.00	2120.00
3	2120.00	254.40	1000.00	3374.40
4	3374.40	404.93	1000.00	4779.33
5	4779.33	573.52	1000.00	6352.85
6	6352.85	762.34	1000.00	8115.19

Example

Example (Ten Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	0.00	0.00	1000.00	1000.00
2	1000.00	120.00	1000.00	2120.00
3	2120.00	254.40	1000.00	3374.40
4	3374.40	404.93	1000.00	4779.33
5	4779.33	573.52	1000.00	6352.85
6	6352.85	762.34	1000.00	8115.19
7	8115.19	973.82	1000.00	10089.01

Example

Example (Ten Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	0.00	0.00	1000.00	1000.00
2	1000.00	120.00	1000.00	2120.00
3	2120.00	254.40	1000.00	3374.40
4	3374.40	404.93	1000.00	4779.33
5	4779.33	573.52	1000.00	6352.85
6	6352.85	762.34	1000.00	8115.19
7	8115.19	973.82	1000.00	10089.01
8	10089.01	1210.68	1000.00	12299.69

Example

Example (Ten Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	0.00	0.00	1000.00	1000.00
2	1000.00	120.00	1000.00	2120.00
3	2120.00	254.40	1000.00	3374.40
4	3374.40	404.93	1000.00	4779.33
5	4779.33	573.52	1000.00	6352.85
6	6352.85	762.34	1000.00	8115.19
7	8115.19	973.82	1000.00	10089.01
8	10089.01	1210.68	1000.00	12299.69
9	12299.69	1475.96	1000.00	14775.65

Example

Example (Ten Deposits)

The investment stage:

Year	Starting Balance	Interest	Investment	Ending Balance
1	0.00	0.00	1000.00	1000.00
2	1000.00	120.00	1000.00	2120.00
3	2120.00	254.40	1000.00	3374.40
4	3374.40	404.93	1000.00	4779.33
5	4779.33	573.52	1000.00	6352.85
6	6352.85	762.34	1000.00	8115.19
7	8115.19	973.82	1000.00	10089.01
8	10089.01	1210.68	1000.00	12299.69
9	12299.69	1475.96	1000.00	14775.65
10	14775.65	1773.08	1000.00	17548.73

Example

Example (Ten Withdrawals)

- Now we begin withdrawing over the next 10 years.
- How much can we withdraw each year?

Example

Example (Ten Withdrawals)

- Now we begin withdrawing over the next 10 years.
- How much can we withdraw each year?

$$M = \frac{P^*}{1 - (1 + r^*)^{-n}}$$

Example

Example (Ten Withdrawals)

- Now we begin withdrawing over the next 10 years.
- How much can we withdraw each year?

$$M = \frac{P^*}{1-(1+r^*)^{-n}} = \frac{(17548.74)(0.12)}{1-(1.12)^{-10}}$$

Example

Example (Ten Withdrawals)

- Now we begin withdrawing over the next 10 years.
- How much can we withdraw each year?

$$\begin{aligned} M &= \frac{P^*}{1-(1+r^*)^{-n}} \\ &= \frac{(17548.74)(0.12)}{1-(1.12)^{-10}} \\ &= 3105.85 \end{aligned}$$

Example

Example (Ten Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	17548.74	2105.85	3105.85	16548.74

Example

Example (Ten Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	17548.74	2105.85	3105.85	16548.74
2	16548.74	1985.85	3105.85	15428.74

Example

Example (Ten Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	17548.74	2105.85	3105.85	16548.74
2	16548.74	1985.85	3105.85	15428.74
3	15428.74	1851.45	3105.85	14174.34

Example

Example (Ten Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	17548.74	2105.85	3105.85	16548.74
2	16548.74	1985.85	3105.85	15428.74
3	15428.74	1851.45	3105.85	14174.34
4	14174.34	1700.92	3105.85	12769.41

Example

Example (Ten Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	17548.74	2105.85	3105.85	16548.74
2	16548.74	1985.85	3105.85	15428.74
3	15428.74	1851.45	3105.85	14174.34
4	14174.34	1700.92	3105.85	12769.41
5	12769.41	1532.33	3105.85	11195.89

Example

Example (Ten Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	17548.74	2105.85	3105.85	16548.74
2	16548.74	1985.85	3105.85	15428.74
3	15428.74	1851.45	3105.85	14174.34
4	14174.34	1700.92	3105.85	12769.41
5	12769.41	1532.33	3105.85	11195.89
6	11195.89	1343.51	3105.85	9433.55

Example

Example (Ten Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	17548.74	2105.85	3105.85	16548.74
2	16548.74	1985.85	3105.85	15428.74
3	15428.74	1851.45	3105.85	14174.34
4	14174.34	1700.92	3105.85	12769.41
5	12769.41	1532.33	3105.85	11195.89
6	11195.89	1343.51	3105.85	9433.55
7	9433.55	1132.03	3105.85	7459.72

Example

Example (Ten Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	17548.74	2105.85	3105.85	16548.74
2	16548.74	1985.85	3105.85	15428.74
3	15428.74	1851.45	3105.85	14174.34
4	14174.34	1700.92	3105.85	12769.41
5	12769.41	1532.33	3105.85	11195.89
6	11195.89	1343.51	3105.85	9433.55
7	9433.55	1132.03	3105.85	7459.72
8	7459.72	895.17	3105.85	5249.04

Example

Example (Ten Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	17548.74	2105.85	3105.85	16548.74
2	16548.74	1985.85	3105.85	15428.74
3	15428.74	1851.45	3105.85	14174.34
4	14174.34	1700.92	3105.85	12769.41
5	12769.41	1532.33	3105.85	11195.89
6	11195.89	1343.51	3105.85	9433.55
7	9433.55	1132.03	3105.85	7459.72
8	7459.72	895.17	3105.85	5249.04
9	5249.04	629.88	3105.85	2773.07

Example

Example (Ten Withdrawals)

The withdrawal stage:

Year	Starting Balance	Interest	Withdrawal	Ending Balance
1	17548.74	2105.85	3105.85	16548.74
2	16548.74	1985.85	3105.85	15428.74
3	15428.74	1851.45	3105.85	14174.34
4	14174.34	1700.92	3105.85	12769.41
5	12769.41	1532.33	3105.85	11195.89
6	11195.89	1343.51	3105.85	9433.55
7	9433.55	1132.03	3105.85	7459.72
8	7459.72	895.17	3105.85	5249.04
9	5249.04	629.88	3105.85	2773.07
10	2773.07	332.77	3105.85	−0.01

Outline

Another Example

Example

- A person earning $48,000 a year invests 5% of his income in a retirement account earning 9% per year for 45 years.
- How much does he have at the end of 45 years?
- He retires and expects to live for 20 more years.
- How much can he withdraw each year for 20 years?

Another Example

Example

- A person earning $48,000 a year invests 5% of his income in a retirement account earning 9% per year for 45 years.
- How much does he have at the end of 45 years?
- He retires and expects to live for 20 more years.
- How much can he withdraw each year for 20 years?
- What if the interest rate had been 15%?

Another Example

Example

- A person earning $48,000 a year invests 5% of his income in a retirement account earning 9% per year for 45 years.
- How much does he have at the end of 45 years?
- He retires and expects to live for 20 more years.
- How much can he withdraw each year for 20 years?
- What if the interest rate had been 15%?
- What if the inflation rate were 3%?

Outline

Assignment

Assignment

- See handout.