Shapley-Shubik Power

Lecture 14
Section 2.3

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Outline

Introduction

In the national political conventions, when the role is called for votes, the state delegations vie for the honor of being the state that puts their candidate “over the top.”

- Does it really matter?

Introduction

In the national political conventions, when the role is called for votes, the state delegations vie for the honor of being the state that puts their candidate “over the top.”

- Does it really matter?
- No.

Outline

Definitions

Definition (Sequential Coalition)

A sequential coalition is an ordered list of all the players.

If there are N players, then there are $N!$ sequential coalitions.

Definition (Pivotal Player)

When counting up the votes, from left to right, in a sequential coalition, the player who *first* makes the total reach the quota is the pivotal player for that sequence.

Definitions

Definition (Sequential Coalition)

A sequential coalition is an ordered list of all the players.

If there are N players, then there are $N!$ sequential coalitions.

Definition (Pivotal Player)

When counting up the votes, from left to right, in a sequential coalition, the player who *first* makes the total reach the quota is the pivotal player for that sequence.

Every sequential coalition will have exactly one pivotal player.

Outline

Listing Permutations

- Given two players, there two permutations:

$$AB, BA.$$

Listing Permutations

- Given two players, there two permutations:

$$AB, BA.$$

- Given three players, there are six permutations:

$$ABC, ACB, BAC, BCA, CAB, CBA.$$

Listing Permutations

- Given two players, there two permutations:

$$AB, BA.$$

- Given three players, there are six permutations:

$$ABC, ACB, BAC, BCA, CAB, CBA.$$

- What about four players?

Listing Permutations

ABCD	*BACD*	*CABD*	*DABC*
ABDC	*BADC*	*CADB*	*DACB*
ACBD	*BCAD*	*CBAD*	*DBAC*
ACDB	*BCDA*	*CBDA*	*DBCA*
ADBC	*BDAC*	*CDAB*	*DCAB*
ADCB	*BDCA*	*CDBA*	*DCBA*

Four players

Listing Permutations

ABCDE	*ADBCE*	*BACDE*	*BDACE*	*CABDE*	*CDABE*	*DABCE*	*DCABE*	*EABCD*	*ECABD*
ABCED	*ADBEC*	*BACED*	*BDAEC*	*CABED*	*CDAEB*	*DABEC*	*DCAEB*	*EABDC*	*ECABD*
ABDCE	*ADCBE*	*BADCE*	*BDCAE*	*CADBE*	*CDBAE*	*DACBE*	*DCBAE*	*EACBD*	*ECBAD*
ABDEC	*ADCEB*	*BADEC*	*BDCEA*	*CADEB*	*CDBEA*	*DACEB*	*DCBEA*	*EACDB*	*ECBDA*
ABECD	*ADEBC*	*BAECD*	*BDEAC*	*CAEBD*	*CDEAB*	*DAEBC*	*DCEAB*	*EADBC*	*ECBAD*
ABEDC	*ADECB*	*BAEDC*	*BDECA*	*CAEDB*	*CDEBA*	*DAECB*	*DCEBA*	*EADCB*	*ECBDA*
ACBDE	*AEBCD*	*BCADE*	*BEACD*	*CBADE*	*CEABD*	*DBACE*	*DEABC*	*EBACD*	*EDABC*
ACBED	*AEBDC*	*BCAED*	*BEADC*	*CBAED*	*CEADB*	*DBAEC*	*DEACB*	*EBADC*	*EDACB*
ACDBE	*AECBD*	*BCDAE*	*BECAD*	*CBDAE*	*CEBAD*	*DBCAE*	*DEBAC*	*EBCAD*	*EDBAC*
ACDEB	*AECDB*	*BCDEA*	*BECDA*	*CBDEA*	*CEBDA*	*DBCEA*	*DEBCA*	*EBCDA*	*EDBCA*
ACEBD	*AEDBC*	*BCEAD*	*BEDAC*	*CBEAD*	*CEDAB*	*DBEAC*	*DECAB*	*EBDAC*	*EDCAB*
ACEDB	*AEDCB*	*BCEDA*	*BEDCA*	*CBEDA*	*CEDBA*	*DBECA*	*DECBA*	*EBDCA*	*EDCBA*

Five players

Six Players

ABCDEF	ACBDEF	ADBCEF	AEBCDF	AFBCDE	BACDEF	BCADEF	BDACEF	BEACDF	BFACDE
ABCDFE	ACBDFE	ADBCFE	AEBCFD	AFBCED	BACDFE	BCADFE	BDACFE	BEACFD	BFACED
ABCEDF	ACBEDF	ADBECF	AEBDCF	AFBDCE	BACEDF	BCAEDF	BDAECF	BEADCF	BFADCE
ABCEFD	ACBEFD	ADBEFC	AEBDFC	AFBDEC	BACEFD	BCAEFD	BDAEFC	BEADFC	BFADEC
ABCFDE	ACBFDE	ADBFCE	AEBFCD	AFBECD	BACFDE	BCAFDE	BDAFCE	BEAFCD	BFAECD
ABCFED	ACBFED	ADBFEC	AEBFDC	AFBEDC	BACFED	BCAFED	BDAFEC	BEAFDC	BFAEDC
ABDCEF	ACDBEF	ADCBEF	AECBDF	AFCBDE	BADCEF	BCDAEF	BDCAEF	BECADF	BFCADE
ABDCFE	ACDBFE	ADCBFE	AECBFD	AFCBED	BADCFE	BCDAFE	BDCAFE	BECAFD	BFCAED
ABDECF	ACDEBF	ADCEBF	AECDBF	AFCDBE	BADECF	BCDEAF	BDCEAF	BECDAF	BFCDAE
ABDEFC	ACDEFB	ADCEFB	AECDFB	AFCDEB	BADEFC	BCDEFA	BDCEFA	BECDFA	BFCDEA
ABDFCE	ACDFBE	ADCFBE	AECFBD	AFCEBD	BADFCE	BCDFAE	BDCFAE	BECFAD	BFCEAD
ABDFEC	ACDFEB	ADCFEB	AECFDB	AFCEDB	BADFEC	BCDFEA	BDCFEA	BECFDA	BFCEDA
ABECDF	ACEBDF	ADEBCF	AEDBCF	AFDBCE	BAECDF	BCEADF	BDEACF	BEDACF	BFDACE
ABECFD	ACEBFD	ADEBFC	AEDBFC	AFDBEC	BAECFD	BCEAFD	BDEAFC	BEDAFC	BFDAEC
ABEDCF	ACEDBF	ADECBF	AEDCBF	AFDCBE	BAEDCF	BCEDAF	BDECAF	BEDCAF	BFDCAE
ABEDFC	ACEDFB	ADECFB	AEDCFB	AFDCEB	BAEDFC	BCEDFA	BDECFA	BEDCFA	BFDCEA
ABEFCD	ACEFBD	ADEFBC	AEDFBC	AFDEBC	BAEFCD	BCEFAD	BDEFAC	BEDFAC	BFDEAC
ABEFDC	ACEFDB	ADEFCB	AEDFCB	AFDECB	BAEFDC	BCEFDA	BDEFCA	BEDFCA	BFDECA
ABFCDE	ACFBDE	ADFBCE	AEFBCD	AFEBCD	BAFCDE	BCFADE	BDFACE	BEFACD	BFEACD
ABFCED	ACFBED	ADFBEC	AEFBDC	AFEBDC	BAFCED	BCFAED	BDFAEC	BEFADC	BFEADC
ABFDCE	ACFDBE	ADFCBE	AEFCBD	AFECBD	BAFDCE	BCFDAE	BDFCAE	BEFCAD	BFECAD
ABFDEC	ACFDEB	ADFCEB	AEFCDB	AFECDB	BAFDEC	BCFDEA	BDFCEA	BEFCDA	BFECDA
ABFECD	ACFEBD	ADFEBC	AEFDBC	AFEDBC	BAFECD	BCFEAD	BDFEAC	BEFDAC	BFEDAC
ABFEDC	ACFEDB	ADFECB	AEFDCB	AFEDCB	BAFEDC	BCFEDA	BDFECA	BEFDCA	BFEDCA

Six players (part 1 of 3 parts)

Six Players

CABDEF	CBADEF	CDABEF	CEABDF	CFABDE	DABCEF	DBACEF	DCABEF	DEABCF	DFABCE
CABDFE	CBADFE	CDABFE	CEABFD	CFABED	DABCFE	DBACFE	DCABFE	DEABFC	DFABEC
CABEDF	CBAEDF	CDAEBF	CEADBF	CFADBE	DABECF	DBAECF	DCAEBF	DEACBF	DFACBE
CABEFD	CBAEFD	CDAEFB	CEADFB	CFADEB	DABEFC	DBAEFC	DCAEFB	DEACFB	DFACEB
CABFDE	CBAFDE	CDAFBE	CEAFBD	CFAEBD	DABFCE	DBAFCE	DCAFBE	DEAFBC	DFAEBC
CABFED	CBAFED	CDAFEB	CEAFDB	CFAEDB	DABFEC	DBAFEC	DCAFEB	DEAFCB	DFAECB
CADBEF	CBDAEF	CDBAEF	CEBADF	CFBADE	DACBEF	DBCAEF	DCBAEF	DEBACF	DFBACE
CADBFE	CBDAFE	CDBAFE	CEBAFD	CFBAED	DACBFE	DBCAFE	DCBAFE	DEBAFC	DFBAEC
CADEBF	CBDEAF	CDBEAF	CEBDAF	CFBDAE	DACEBF	DBCEAF	DCBEAF	DEBCAF	DFBCAE
CADEFB	CBDEFA	CDBEFA	CEBDFA	CFBDEA	DACEFB	DBCEFA	DCBEFA	DEBCFA	DFBCEA
CADFBE	CBDFAE	CDBFAE	CEBFAD	CFBEAD	DACFBE	DBCFAE	DCBFAE	DEBFAC	DFBEAC
CADFEB	CBDFEA	CDBFEA	CEBFDA	CFBEDA	DACFEB	DBCFEA	DCBFEA	DEBFCA	DFBECA
CAEBDF	CBEADF	CDEABF	CEDABF	CFDABE	DAEBCF	DBEACF	DCEABF	DECABF	DFCABE
CAEBFD	CBEAFD	CDEAFB	CEDAFB	CFDAEB	DAEBFC	DBEAFC	DCEAFB	DECAFB	DFCAEB
CAEDBF	CBEDAF	CDEBAF	CEDBAF	CFDBAE	DAECBF	DBECAF	DCEBAF	DECBAF	DFCBAE
CAEDFB	CBEDFA	CDEBFA	CEDBFA	CFDBEA	DAECFB	DBECFA	DCEBFA	DECBFA	DFCBEA
CAEFBD	CBEFAD	CDEFAB	CEDFAB	CFDEAB	DAEFBC	DBEFAC	DCEFAB	DECFAB	DFCEAB
CAEFDB	CBEFDA	CDEFBA	CEDFBA	CFDEBA	DAEFCB	DBEFCA	DCEFBA	DECFBA	DFCEBA
CAFBDE	CBFADE	CDFABE	CEFABD	CFEABD	DAFBCE	DBFACE	DCFABE	DEFABC	DFEABC
CAFBED	CBFAED	CDFAEB	CEFADB	CFEADB	DAFBEC	DBFAEC	DCFAEB	DEFACB	DFEACB
CAFDBE	CBFDAE	CDFBAE	CEFBAD	CFEBAD	DAFCBE	DBFCAE	DCFBAE	DEFBAC	DFEBAC
CAFDEB	CBFDEA	CDFBEA	CEFBDA	CFEBDA	DAFCEB	DBFCEA	DCFBEA	DEFBCA	DFEBCA
CAFEBD	CBFEAD	CDFEAB	CEFDAB	CFEDAB	DAFEBC	DBFEAC	DCFEAB	DEFCAB	DFECAB
CAFEDB	CBFEDA	CDFEBA	CEFDBA	CFEDBA	DAFECB	DBFECA	DCFEBA	DEFCBA	DFECBA

Six players (part 2 of 3 parts)

Six Players

EABCDF	EBACDF	ECABDF	EDABCF	EFABCD	FABCDE	FBACDE	FCABDE	FDABCE	FEABCD
EABCFD	EBACFD	ECABFD	EDABFC	EFABDC	FABCED	FBACED	FCABED	FDABEC	FEABDC
EABDCF	EBADCF	ECADBF	EDACBF	EFACBD	FABDCE	FBADCE	FCADBE	FDACBE	FEACBD
EABDFC	EBADFC	ECADFB	EDACFB	EFACDB	FABDEC	FBADEC	FCADEB	FDACEB	FEACDB
EABFCD	EBAFCD	ECAFBD	EDAFBC	EFADBC	FABECD	FBAECD	FCAEBD	FDAEBC	FEADBC
EABFDC	EBAFDC	ECAFDB	EDAFCB	EFADCB	FABEDC	FBAEDC	FCAEDB	FDAECB	FEADCB
EACBDF	EBCADF	ECBADF	EDBACF	EFBACD	FACBDE	FBCADE	FCBADE	FDBACE	FEBACD
EACBFD	EBCAFD	ECBAFD	EDBAFC	EFBADC	FACBED	FBCAED	FCBAED	FDBAEC	FEBADC
EACDBF	EBCDAF	ECBDAF	EDBCAF	EFBCAD	FACDBE	FBCDAE	FCBDAE	FDBCAE	FEBCAD
EACDFB	EBCDFA	ECBDFA	EDBCFA	EFBCDA	FACDEB	FBCDEA	FCBDEA	FDBCEA	FEBCDA
EACFBD	EBCFAD	ECBFAD	EDBFAC	EFBDAC	FACEBD	FBCEAD	FCBEAD	FDBEAC	FEBDAC
EACFDB	EBCFDA	ECBFDA	EDBFCA	EFBDCA	FACEDB	FBCEDA	FCBEDA	FDBECA	FEBDCA
EADBCF	EBDACF	ECDABF	EDCABF	EFCABD	FADBCE	FBDACE	FCDABE	FDCABE	FECABD
EADBFC	EBDAFC	ECDAFB	EDCAFB	EFCADB	FADBEC	FBDAEC	FCDAEB	FDCAEB	FECADB
EADCBF	EBDCAF	ECDBAF	EDCBAF	EFCBAD	FADCBE	FBDCAE	FCDBAE	FDCBAE	FECBAD
EADCFB	EBDCFA	ECDBFA	EDCBFA	EFCBDA	FADCEB	FBDCEA	FCDBEA	FDCBEA	FECBDA
EADFBC	EBDFAC	ECDFAB	EDCFAB	EFCDAB	FADEBC	FBDEAC	FCDEAB	FDCEAB	FECDAB
EADFCB	EBDFCA	ECDFBA	EDCFBA	EFCDBA	FADECB	FBDECA	FCDEBA	FDCEBA	FECDBA
EAFBCD	EBFACD	ECFABD	EDFABC	EFDABC	FAEBCD	FBEACD	FCEABD	FDEABC	FEDABC
EAFBDC	EBFADC	ECFADB	EDFACB	EFDACB	FAEBDC	FBEADC	FCEADB	FDEACB	FEDACB
EAFCBD	EBFCAD	ECFBAD	EDFBAC	EFDBAC	FAECBD	FBECAD	FCEBAD	FDEBAC	FEDBAC
EAFCDB	EBFCDA	ECFBDA	EDFBCA	EFDBCA	FAECDB	FBECDA	FCEBDA	FDEBCA	FEDBCA
EAFDBC	EBFDAC	ECFDAB	EDFCAB	EFDCAB	FAEDBC	FBEDAC	FCEDAB	FDECAB	FEDCAB
EAFDCB	EBFDCA	ECFDBA	EDFCBA	EFDCBA	FAEDCB	FBEDCA	FCEDBA	FDECBA	FEDCBA

Six players (part 3 of 3 parts)

Seven Players

- Click here for seven players.

Outline

Example

Example

Consider the situation $[4 : 3, 2, 1]$. List all sequential coalitions and determine the pivotal player for each one.

Sequential Coalition	Vote Totals			Pivotal Player
	1st	2nd	3rd	
$\{A, B, C\}$				
$\{A, C, B\}$				
$\{B, A, C\}$				
$\{B, C, A\}$				
$\{C, A, B\}$				
$\{C, B, A\}$				

Example

Example

Consider the situation $[4 : 3, 2, 1]$. List all sequential coalitions and determine the pivotal player for each one.

Sequential Coalition	Vote Totals			Pivotal Player
	1st	2nd	3rd	
$\{A, B, C\}$	3			
$\{A, C, B\}$	3			
$\{B, A, C\}$	2			
$\{B, C, A\}$	2			
$\{C, A, B\}$	1			
$\{C, B, A\}$	1			

Example

Example

Consider the situation $[4 : 3, 2, 1]$. List all sequential coalitions and determine the pivotal player for each one.

Sequential Coalition	Vote Totals			Pivotal Player
	1st	2nd	3rd	
$\{A, B, C\}$	3	5		B
$\{A, C, B\}$	3	4		C
$\{B, A, C\}$	2	5		A
$\{B, C, A\}$	2	3		
$\{C, A, B\}$	1	4		A
$\{C, B, A\}$	1	3		

Example

Example

Consider the situation $[4 : 3, 2, 1]$. List all sequential coalitions and determine the pivotal player for each one.

Sequential Coalition	Vote Totals			Pivotal Player
	1st	2nd	3rd	
$\{A, B, C\}$	3	5	6	B
$\{A, C, B\}$	3	4	6	C
$\{B, A, C\}$	2	5	6	A
$\{B, C, A\}$	2	3	6	A
$\{C, A, B\}$	1	4	6	A
$\{C, B, A\}$	1	3	6	A

Shapley-Shubik Power

Definition (Pivotal Count)

A player's pivotal count is the number of sequential coalitions in which he is the pivotal player.

In the previous example, the pivotal counts are 4, 1, 1.

Definition (Shapley-Shubik Power Index)

The Shapley-Shubik power index (SSPI) for a player is that player's pivotal count divided by the total of the counts.

Definition (Shapley-Shubik Power Distribution)

The Shapley-Shubik power distribution is the set of SSPI's for all the players.

Outline

Example

Example

- Find the Shapley-Shubik power distribution for $[14 : 9, 8, 5, 4]$.
- Compare it to the Banzhaf power distribution.

Example

Example

ABCD 9854	BACD 8954	CABD 5984	DABC 4985
ABDC 9845	BADC 8945	CADB 5948	DACB 4958
ACBD 9584	BCAD 8594	CBAD 5894	DBAC 4895
ACDB 9548	BCDA 8549	CBDA 5849	DBCA 4859
ADBC 9485	BDAC 8495	CDAB 5498	DCAB 4598
ADCB 9458	BDCA 8459	CDBA 5489	DCBA 4589

Example

Example

ABCD 9854	BACD 8954	CABD 5984	DABC 4985
ABDC 9845	BADC 8945	CADB 5948	DACB 4958
ACBD 9584	BCAD 8594	CBAD 5894	DBAC 4895
ACDB 9548	BCDA 8549	CBDA 5849	DBCA 4859
ADBC 9485	BDAC 8495	CDAB 5498	DCAB 4598
ADCB 9458	BDCA 8459	CDBA 5489	DCBA 4589

Example

Example

- Find the Shapley-Shubik power distribution for $[34 : 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1]$.
- Compare it to the Banzhaf power distribution.

Outline

The Electoral College

- In the electoral college, each state get a number of votes equal to its number of representatives plus its number of senators.
- “Each State shall appoint, in such Manner as the Legislature thereof may direct, a Number of Electors, equal to the whole Number of Senators and Representatives to which the State may be entitled in the Congress.”
- The number of representatives is proportional to the state’s population, so why add the two senators?

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- “Each State shall appoint, in such Manner as the Legislature thereof may direct, a Number of Electors, equal to the whole Number of Senators and Representatives to which the State may be entitled in the Congress.”
- The number of representatives is proportional to the state’s population, so why add the two senators?
- It was to give the smaller states more power relative to the larger states.

The Election of 1788

- In 1788, only 11 states participated in the presidential election.
- They cast a total of 81 electoral votes, 41 needed to win.

VA	12
MA	10
PA	10
MD	8
NY	8
CT	7

SC	7
NJ	6
GA	5
NH	5
DE	3

- Determine the power of each state.

The Election of 1788

- What if the senators were not counted?
- They would cast a total of 59 electoral votes, 30 needed to win.

VA	10
MA	8
PA	8
MD	6
NY	6
CT	5

SC	5
NJ	4
GA	3
NH	3
DE	1

- Determine the power of each state.

The Election of 1792

- In 1792, 15 states participated in the presidential election.
- They cast a total of 135 electoral votes, 68 needed to win.

VA	21
MA	16
PA	15
NY	12
NC	12

MD	10
CT	9
SC	8
NJ	7
NH	6

GA	4
KY	4
RI	4
VT	4
DE	3

- Determine the power of each state.

The Election of 1788

- What if the senators were not counted?
- They would cast a total of 105 electoral votes, 53 needed to win.

VA	19
MA	14
PA	13
NY	10
NC	10

MD	8
CT	7
SC	6
NJ	5
NH	4

GA	2
KY	2
RI	2
VT	2
DE	1

- Determine the power of each state.

Outline

Assignment

Assignment

- Ch. 2: 27, 28, 29, 30, 32, 33; 56.