Direct Proof – Universal Statements

Lecture 13
Section 4.1

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Wed, Feb 6, 2013

Outline

Proving Universal Statements

- A universal statement is generally of the form

$$\forall x \in D, P(x) \to Q(x).$$

- Use the method of generalizing from the generic particular.
  - Select an *arbitrary* $x \in D$ (generic particular).
  - Suppose that $P(x)$ is true (hypothesis).
  - Argue that $Q(x)$ is true (conclusion).

Outline

Example

Theorem

If n is an odd integer, then $n^3 - n$ is a multiple of 12.

Proof.

- Let n be an odd integer.
- Then $n = 2k + 1$ for some integer k.
- Then

$$\begin{aligned} n^3 - n &= (2k+1)^3 - (2k+1) \\ &= (2k+1)\left[(2k+1)^2 - 1\right] \\ &= (2k+1)(4k^2 + 4k) \\ &= 4k(2k+1)(k+1). \end{aligned}$$

□

Example

Proof.

- Consider cases. k is either (1) a multiple of 3, or (2) 1 greater than a multiple of 3, or (3) 1 less than a multiple of 3.
- That is, $k = 3m$, $k = 3m + 1$, or $k = 3m - 1$ for some integer m.

Case 1: Suppose $k = 3m$. Then

$$\begin{aligned} n^3 - n &= 4(3m)(6m+1)(3m+1) \\ &= 12m(6m+1)(3m+1). \end{aligned}$$

Case 2: Suppose $k = 3m + 1$. Then

$$\begin{aligned} n^3 - n &= 4(3m+1)(6m+3)(3m+2) \\ &= 12(3m+1)(2m+1)(3m+2). \end{aligned}$$

□

Example

Proof.

Case 3: Suppose $k = 3m - 1$. Then

$$\begin{aligned} n^3 - n &= 4(3m-1)(6m-1)(3m) \\ &= 12(3m-1)(6m-1)(m). \end{aligned}$$

- Thus, $n^3 - n$ is a multiple of 12.
- Is the same true if n is even?
- Is the same always false when n is even?

□

Example

Alternate Proof.

- Note that

$$n^3 - n = (n-1)(n)(n+1),$$

  the product of three consecutive integers.
- Therefore, one of them must be a multiple of 3.
- Since n is odd, then $n-1$ and $n+1$ are even.
- Therefore, $n^3 - n$ is a multiple of 12.

□

Outline

Example

Theorem

For all $x, y \in \mathbb{R}$*,*

$$x^2 + y^2 \geq 2xy.$$

Proof.

- Let $x, y \in \mathbb{R}$. Then

$$\begin{aligned} x^2 + y^2 &\geq 2xy \\ x^2 - 2xy + y^2 &\geq 0 \\ (x - y)^2 &\geq 0. \end{aligned}$$

- We know that $(x - y)^2$ must be ≥ 0, therefore, the theorem is true.

□

Example

- What is the flaw in the logic of that “proof?”
- How do we correct it?

Example

Proof.

- Let $x, y \in \mathbb{R}$.
- We know that $(x - y)^2 \geq 0$.
- Therefore,

$$\begin{aligned} x^2 - 2xy + y^2 &\geq 0 \\ x^2 + y^2 &\geq 2xy. \end{aligned}$$

$\square$

Outline

Example

Theorem

There is no solution in integers to the equation

$$2^n = n.$$

- This is the same as saying that

$$\forall n \in \mathbb{N}, 2^n \neq n.$$

Example

- Certainly, if $n < 0$, then $2^n \neq n$.
- Try a few nonnegative values of n:

$$2^0 = 1 \neq 0.$$
$$2^1 = 2 \neq 1.$$
$$2^2 = 4 \neq 2.$$
$$2^3 = 8 \neq 3.$$

- Do we get any insight to a general proof?

Example

- Clearly, the statement is true for $n = 0, 1, 2, 3$.
- Suppose that $n > 3$.
- Then

$$\begin{aligned} 2^n &= 2 \cdot 2^{n-1} \\ &= 2^{n-1} + 2^{n-1} \\ &> (n-1) + 1 \\ &= n. \end{aligned}$$

- Thus, $2^n \neq n$.

Example

Theorem

There is no solution in real numbers to the equation

$$2^x = x.$$

- We can prove this using calculus.
- Let $f(x) = 2^x - x$.
- Use calculus to find the minimum value of f.
- The minimum value turns out to be a positive number.
- Thus, $f(x) \neq 0$ for all $x \in \mathbb{R}$.
- (This proof covers the case for integers as well.)

Outline

Example: Proof by Cases

Theorem

There is no solution in integers to the equation

$$x^2 - y^2 = 10^{10^{10}} + 2.$$

However, there is a solution to the equation

$$x^2 - y^2 = 10^{10^{10}} + 1.$$

-

Example: Proof by Cases

Proof.

- There are four possibilities:
  (1) x and y are both even.
  (2) x is even and y is odd.
  (3) x is odd and y is even.
  (4) x and y are both odd.

□

Example: Proof by Cases

Proof.

Case (1): Suppose that x and y are both even.

- Then $x = 2m$ and $y = 2n$ for some integers m and n.
- Then

$$\begin{aligned} x^2 - y^2 &= 4m^2 - 4n^2 \\ &= 4(m^2 - n^2), \end{aligned}$$

  which must be a multiple of 4.
- However, $10^{10^{10}} + 2$ is not a multiple of 4.
- Therefore, $x^2 - y^2 \neq 10^{10^{10}} + 2$.

The other cases are similar.

□

Proof.

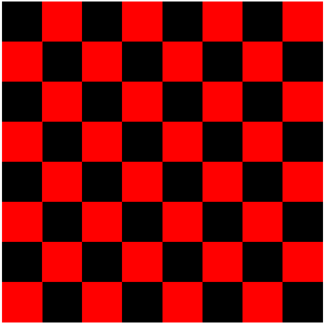
- Let $a = 10^{10^{10}}$.
- Let $x = \frac{a}{2} + 1$ and $y = \frac{a}{2}$.
- Then

$$\begin{aligned} x^2 - y^2 &= \left(\frac{a^2}{4} + a + 1\right) - \frac{a^2}{4} \\ &= a + 1 \\ &= 10^{10^{10}} + 1. \end{aligned}$$

□

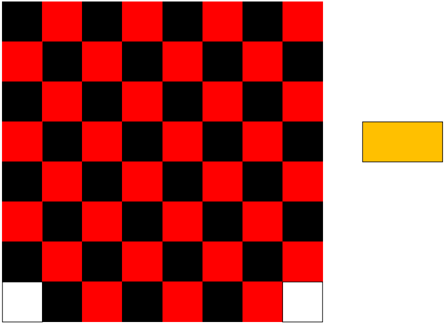
Outline

Proving an Existential Statement



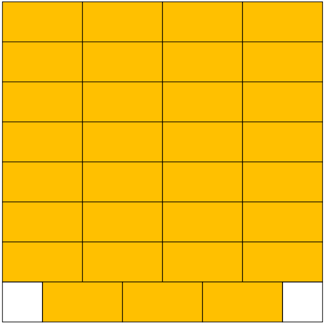
A checkerboard

Proving an Existential Statement



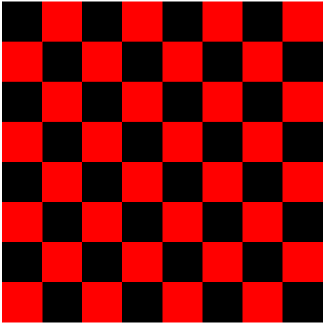
Remove two squares and cover board with 1×2 blocks

Proving an Existential Statement



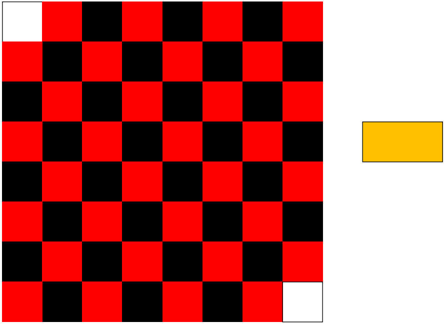
It can be done

Disproving an Existential Statement



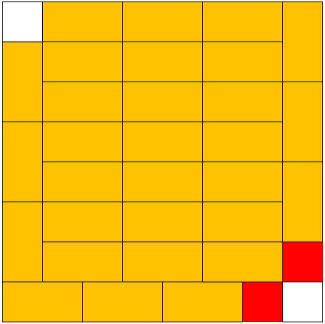
A checkerboard

Disproving an Existential Statement



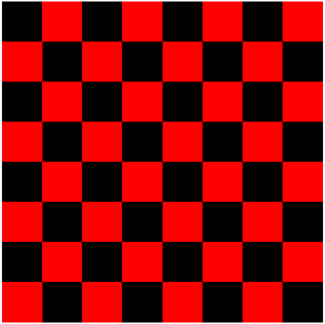
Remove two squares and cover board with 1×2 blocks

Disproving an Existential Statement



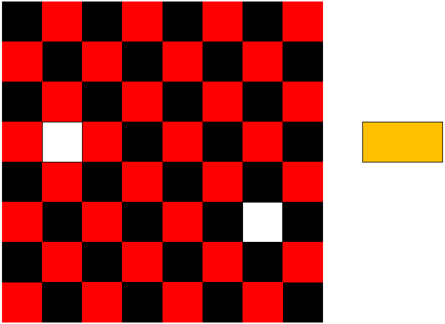
It cannot be done

A Universal and Existential Statement



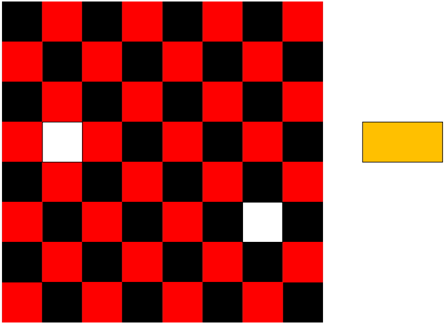
Remove any two squares of *opposite* color

A Universal and Existential Statement



Can the board necessarily be covered?

A Universal and Existential Statement



Is that true for any $2n \times 2n$ board?

Outline

Assignment

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- Read Section 4.1, pages 145 - 160.
- Exercises 20, 21, 25, 28, 30, 37, 52, 53, 55, 58, page 161.

Collected Homework 3

Collected Homework 3

- Exercises 18, 29, page 106.
- Exercises 17, 25, page 115.
- Exercises 19, 56, page 129.
- Exercises 10, 18, page 161.