Mathematical Induction – Introduction
Lecture 21
Section 5.2

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1 The Principle

2 The Method

3 Examples

4 Assignment
Let $P(n)$ be a predicate defined for all integers $n \geq 0$.

If the following two statements are true

- $P(0)$,
- For all $k \geq 0$, if $P(k)$, then $P(k + 1)$,

then the statement

- For all integers $n \geq 0$, $P(n)$.

is true.
The Principle

- The first statement shows that $P(0)$ is true.
- The second statement shows that $P(0) \rightarrow P(1)$.
- Now that we have $P(1)$, the second statement shows that $P(1) \rightarrow P(2)$.
- Now that we have $P(2)$, the second statement shows that $P(1) \rightarrow P(3)$.
- And so on.
- Therefore, we have $P(n)$ for all $n \geq 0$. 
The Principle
The Principle

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The Principle
The Method

The basic step.
- Choose a starting point \( a \), typically 0 or 1.
- Prove \( P(n) \) for that starting point \( a \), e.g., prove \( P(0) \) or \( P(1) \).

The inductive step.
- Suppose that \( P(k) \) is true for some \( k \geq a \).
- Prove that it follows that \( P(k + 1) \) must be true.

Conclude that \( P(n) \) is true for all \( n \geq a \).
1. The Principle
2. The Method
3. Examples
4. Assignment
Theorem

Let $n \geq 0$. Then

$$\sum_{i=0}^{n} i = \frac{n(n + 1)}{2}.$$
Proof.

- The basic step:
  - When $n = 0$, we have

\[ \sum_{i=0}^{n} i = \sum_{i=0}^{0} i = 0, \]

and

\[ \frac{n(n + 1)}{2} = \frac{0 \cdot 1}{2} = 0. \]

- Therefore, the statement is true when $n = 0$. 
Proof.

The basic step:

- When $n = 0$, we have

\[
\sum_{i=0}^{n} i = \sum_{i=0}^{0} i = 0,
\]

and

\[
\frac{n(n + 1)}{2} = \frac{0 \cdot 1}{2} = 0.
\]

Therefore, the statement is true when $n = 0$. 
The inductive step:

Suppose that the statement is true when \( n = k \), for some integer \( k \geq 0 \).

That is, suppose that \( \sum_{i=0}^{k} i = \frac{k(k+1)}{2} \) for some integer \( k \geq 0 \).

Then

\[
\sum_{i=0}^{k+1} i = \left( \sum_{i=0}^{k} i \right) + (k + 1)
\]

\[
= \frac{k(k+1)}{2} + (k + 1)
\]

\[
= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}
\]

\[
= \frac{(k + 1)(k + 2)}{2}.
\]
Proof.

- The inductive step:
  - Suppose that the statement is true when $n = k$, for some integer $k \geq 0$.
  - That is, suppose that $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$ for some integer $k \geq 0$.
  - Then

\[
\sum_{i=0}^{k+1} i = \left( \sum_{i=0}^{k} i \right) + (k + 1)
\]
\[
= \frac{k(k+1)}{2} + (k + 1)
\]
\[
= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}
\]
\[
= \frac{(k+1)(k+2)}{2}.
\]
Proof.

- The inductive step continued:
  - Therefore, the statement is true when \( n = k + 1 \).
- Therefore, the statement is true for all \( n \geq 0 \).
Proof.

- The inductive step continued:
  - Therefore, the statement is true when $n = k + 1$.
  - Therefore, the statement is true for all $n \geq 0$. 
Theorem

Let \( n \geq 4 \) be an integer. Then \( n \)¢ can be obtained using only 2¢ and 5¢ coins.

- The predicate \( P(n) \) is “\( n \)¢ can be obtained using only 2¢ and 5¢ coins.”
- The starting point is 4.
Proof.

- The basic step:
  - When $n = 4$, we have $4¢ = 2¢ + 2¢$.
  - Therefore, the statement is true when $n = 4$. 
Proof.

- The inductive step:
  - Suppose that the statement is true when \( n = k \), for some integer \( k \geq 4 \).
  - That is, suppose that \( k \)¢ can be obtained using only 2¢ and 5¢ coins for some integer \( k \geq 4 \).
  - We consider 2 cases:
    - Case 1: \( k \)¢ uses a 5¢ coin.
    - Case 2: \( k \)¢ does not use a 5¢ coin.
Proof.
The inductive step continued:

- Case 1: Suppose that $k¢$ uses a 5¢ coin.
  - Then replace it with three 2¢ coins to make $(k + 1)¢$.
- Case 2: Suppose that $k¢$ does not use a 5¢ coin.
  - Then it must use at least two 2¢ coins.
  - Replace them with one 5¢ coin to make $(k + 1)¢$. 
Proof.

- The inductive step continued:
  - Therefore, the statement is true when $n = k + 1$.
  - Therefore, the statement is true for all $n \geq 4$. 
Sums of Squares

- Find and verify a formula for

\[ \sum_{i=1}^{n} i^2. \]

- Why might we guess that the formula is a cubic polynomial?
Let us guess that

$$\sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d,$$

for some real numbers $a, b, c, d$.

How do we find $a, b, c, d$?
Sums of Squares

We know that

\[ 1^2 = 1, \]
\[ 1^2 + 2^2 = 5, \]
\[ 1^2 + 2^2 + 3^2 = 14, \]
\[ 1^2 + 2^2 + 3^2 + 4^2 = 30. \]

Substitute these values and solve for \( a, b, c, d \).
Sums of Squares

- Solve the system of equations

\[
\begin{align*}
    a + b + c + d &= 1, \\
    8a + 4b + 2c + d &= 5, \\
    27a + 9b + 3c + d &= 14, \\
    64a + 16b + 4c + d &= 30.
\end{align*}
\]

- We get \( a = \frac{1}{3}, \ b = \frac{1}{2}, \ c = \frac{1}{6}, \) and \( d = 0. \)
Sums of Squares

Theorem

For any positive integer $n$,

$$
\sum_{i=1}^{n} i^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n
$$

$$
= \frac{n(n + 1)(2n + 1)}{6}.
$$
Proof.

- Let \( n = 1 \).
- Then

\[
\sum_{i=1}^{1} i^2 = 1^2 = 1
\]

- and

\[
\frac{n(n+1)(2n+1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 1.
\]

- Therefore, the statement is true when \( n = 1 \).
Proof.

- Suppose that the statement is true when \( n = k \), for some integer \( k \geq 1 \).
- That is, suppose that

\[
\sum_{i=1}^{k} i^2 = \frac{k(k + 1)(2k + 1)}{6}.
\]
Proof.

Then

\[
\sum_{i=1}^{k+1} i^2 = \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2
\]

\[
= \frac{(k + 1)[k(2k + 1) + 6(k + 1)]}{6}
\]

\[
= \frac{(k + 1)(2k^2 + 7k + 6)}{6}
\]

\[
= \frac{(k + 1)(k + 2)(2k + 3)}{6}.
\]
Proof.

- Therefore, the statement is true when \( n = k + 1 \).
- Therefore, the statement is true for all \( n \geq 1 \).
Assignment

- Read Section 5.2, pages 244 - 256.
- Exercises 2, 4, 5, 7, 14, 26, 29, page 256.