Solving Recursive Sequences by Iteration

Lecture 25
Section 5.7

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1 Solving Recursive Relations by Iteration
- The Towers of Hanoi
- Another Example
- Annuities

2 Assignment

Outline

1. Solving Recursive Relations by Iteration
   - The Towers of Hanoi
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The Towers of Hanoi

- In the Towers of Hanoi problem, we found the recursive sequence to be $m_1 = 1$ and $m_n = 2m_{n-1} + 1$.
- We used it to calculate the first few terms: 1, 3, 7, 15.
- A nonrecursive formula for m_n would allow us to calculate m_n directly for an n.
- How do we find a nonrecursive formula for m_n?

The Towers of Hanoi

- The method of iteration is to apply the recursive formula repeatedly, but postpone most of the calculations, until we see a pattern develop.

$$
\begin{aligned}
m_1 &= 1. \\
m_2 &= 2 + 1. \\
m_3 &= 2(2 + 1) + 1 \\
&= 2^2 + 2 + 1. \\
m_4 &= 2(2^2 + 2 + 1) + 1 \\
&= 2^3 + 2^2 + 2 + 1.
\end{aligned}
$$

- The pattern is clear:

$$m_n = 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1.$$

- This is a sum of a geometric series, for which the formula $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$ applies.
- Thus,

$$\begin{aligned} m_n &= \frac{2^n - 1}{2 - 1} \\ &= 2^n - 1. \end{aligned}$$

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Example

- Define a sequence $\{a_n\}$ by

$$a_0 = 1,$$
$$a_n = 5a_{n-1} - 2,$$

  for all $n \geq 1$.
- Use the method of iteration to find a nonrecursive formula.

Example

- Write expressions for the first few terms, until a clear pattern develops.

$$
\begin{aligned}
a_0 &= 1. \\
a_1 &= 5 - 2. \\
a_2 &= 5(5 - 2) - 2 \\
&= 5^2 - 2 \cdot 5 - 2. \\
a_3 &= 5(5^2 - 2 \cdot 5 - 2) - 2 \\
&= 5^3 - 2 \cdot 5^2 - 2 \cdot 5 - 2. \\
a_4 &= 5(5^3 - 2 \cdot 5^2 - 2 \cdot 5 - 2) - 2 \\
&= 5^4 - 2 \cdot 5^3 - 2 \cdot 5^2 - 2 \cdot 5 - 2.
\end{aligned}
$$

Example

- Again, the pattern is clear:

$$\begin{aligned} a_n &= 5^n - 2\left(5^{n-1} + \cdots + 5^2 + 5 + 1\right) \\ &= 5^n - 2\left(\frac{5^n - 1}{5 - 1}\right) \\ &= 5^n - \frac{5^n - 1}{2} \\ &= \tfrac{1}{2}(5^n + 1). \end{aligned}$$

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Annuities

- An annuity is a sum of money from which we can draw over a period of time.
- Suppose the sum is earning 4% interest annually and a person deposits \$5,000 annually for 40 years.
- How much will he have at retirement?

Annuities

- Let A_n be the amount in the account after n years.
- If we deposit \$5,000 at the *end* of each year, then $A_0 = 0$ and $A_1 = 5000$.
- At the end of each subsequent year, we earn 4% interest on the balance and then deposit another \$5,000.
- The recursive formula is

$$A_n = 0.04A_{n-1} + 5000.$$

Annuities

- We calculate

$$\begin{aligned} A_2 &= 1.04 \cdot 5000 + 5000 \\ &= 5000\,(1.04 + 1)\,. \\ A_3 &= 1.04 \cdot 5000\,(1.04 + 1) + 5000 \\ &= 5000\left(1.04^2 + 1.04 + 1\right). \\ A_4 &= 1.04 \cdot 5000\left(1.04^2 + 1.04 + 1\right) + 5000 \\ &= 5000\left(1.04^3 + 1.04^2 + 1.04 + 1\right). \end{aligned}$$

Annuities

- Clearly,

$$A_n = 5000\left(1.04^{n-1} + \cdots + 1.04^2 + 1.04 + 1\right).$$

- So,

$$\begin{aligned} A_n &= 5000\left(\frac{1.04^n - 1}{1.04 - 1}\right) \\ &= 5000\left(\frac{1.04^n - 1}{0.04}.\right) \end{aligned}$$

- Generalize this to an interest rate of r, a deposit of P, and a period of n years.

Annuities

- Given that the person has accumulated A dollars in his annuity, how much can he withdraw each year for m years of retirement if the balance continues to earn an interest rate of r?

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- Read Section 5.7, pages 304 - 314.
- Exercises 2, 5, 7, 22, 23, 25, 26, 27, 52, page 314.