Three Properties

Closures

Assignment
Outline

1. Three Properties
2. Closures
3. Assignment
Definition (Reflexive, Symmetric, Transitive)

Let $R$ be a relation on a set $A$. Then

- $R$ is **reflexive** if, for all $x \in A$, $(x, x) \in R$.
- $R$ is **symmetric** if, for all $x, y \in A$,
  
  $$ (x, y) \in R \rightarrow (y, x) \in R. $$

- $R$ is **transitive** if, for all $x, y, z \in A$,
  
  $$ ((x, y) \in R \text{ and } (y, z) \in R) \rightarrow (x, z) \in R. $$
Three Properties

- **Reflexivity**: Every element of the set $A$ has the relation to itself.
- **Symmetry**: If any element $x \in A$ has the relation to some element $y \in A$, then $y$ has the same relation to $x$.
- **Transitivity**: If any element $x \in A$ has the relation to some element $y \in A$ and that element $y$ has the same relation to some element $z \in A$, then $x$ has that relation to $z$. 
Examples

- Define a relation $R$ on $\mathbb{Z}$ by $(x, y) \in R$ if $5 \mid (x - y)$.
- Show that $R$ is reflexive, symmetric, and transitive.
Examples

Which of the following relations are reflexive? symmetric? transitive?

- $a \mid b$ on $\mathbb{Z}$.
- $\gcd(a, b) > 1$ on $\mathbb{Z}$.
- $x \times y < 0$ on $\mathbb{R}$.
- $A \subseteq B$ on $\mathcal{P}(X)$.
- $p \rightarrow q$ on a set of statements.
- $p \land q \equiv p$ on a set of statements.
Outline

1. Three Properties
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The Reflexive Closure

Definition (Reflexive closure)

Let $A$ be a set and let $R$ be a relation on $A$. The reflexive closure of $R$, denoted $R^r$, is

$$R \cup \{(a, a) \mid a \in A\}.$$

- $R^r$ is the smallest relation on $A$ that contains $R$ and is reflexive.
Add loops to get the graph of the reflexive closure.
Examples

- What is the reflexive closure of $<$ on $\mathbb{R}$?
- What is the reflexive closure of $\subset$ on $\mathcal{P}(X)$?
- What is the reflexive closure of $\equiv \mod n$ on $\mathbb{Z}$.
The Symmetric Closure

**Definition (Symmetric closure)**

Let $A$ be a set and let $R$ be a relation on $A$. The *symmetric closure* of $R$, denoted $R^s$, is

$$R^s = R \cup \{(b, a) \mid (a, b) \in R\}.$$ 

- $R^s$ is the smallest relation on $A$ that contains $R$ and is symmetric.
Make every arrow double-ended to get the graph of the symmetric closure.
Examples

- What is the symmetric closure of $<$ on $\mathbb{R}$?
- What is the symmetric closure of $\subset$ on $\mathcal{P}(X)$?
- What is the symmetric closure of $\equiv \mod n$ on $\mathbb{Z}$. 
The Transitive Closure

**Definition (Transitive closure)**
Let $A$ be a set and let $R$ be a relation on $A$. The **transitive closure of $R$**, denoted $R^t$, is the smallest subset of $A \times A$ that contains $R$ and is transitive.

- To find the transitive closure of a relation requires an algorithm.
- It is not enough to define

$$R^t = R \cup \{(a, c) \mid (a, b), (b, c) \in R\}.$$  

- Why not?
Given a relation $R$ on a set $A$, the following procedure will produce $R^t$.

1. Let $n = 1$ and let $R_0 = \emptyset$ and $R_1 = R$.
2. While $R_n \neq R_{n-1}$, do the following.
   - Let $R_{n+1} = R_n \cup \{(a, c) \mid (a, b), (b, c) \in R_n\}$.
   - $n \leftarrow n + 1$. 

The Transitive Closure
Connect all back-to-back arrows.
Connect all back-to-back arrows again.
Examples

- What is the transitive closure of \((a, b) \in R\) on \(\mathbb{Z}\) defined by \(b = a + 1\)?
- What is the transitive closure of \((A, B) \in R\) on \(\mathcal{P}(\{1, 2, 3\})\) defined by \(A \cap B \neq \emptyset\)?
- What is the transitive closure of \((A, B) \in R\) on \(\mathcal{P}(\mathbb{Z})\) defined by \(|A - B| < \infty\).
We can do all three closures at the same time.
Define $R$ on $\mathbb{Z}$ by $(a, b) \in R$ if $a - b = 2$.
Find the reflexive, symmetric, and transitive closure of $R$. 
Assignment

- Read Sections 8.2, pages 449 - 457.
Collected Homework 10

- Exercises 11, 17, page 426.
- Exercises 9, 12, 15, page 439.
- Exercises 11, 14, page 448.