Direct Proof – Floor and Ceiling

Lecture 17
Section 4.5

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1. The Floor and Ceiling Functions
2. Theorems
3. Applications
4. Assignment

Outline

1. The Floor and Ceiling Functions
2. Theorems
3. Applications
4. Assignment

The Floor and Ceiling Functions

Definition (The Floor Function)

Let $x \in \mathbb{R}$. Define the floor function, denoted $\lfloor x \rfloor$, to be the unique integer n such that

$$n \leq x < n + 1.$$

Definition (The Ceiling Function)

Let $x \in \mathbb{R}$. Define the ceiling function, denoted $\lceil x \rceil$, to be the unique integer n such that

$$n - 1 < x \leq n.$$

Examples

- For example,
  - $\lfloor 3.8 \rfloor = 3$
  - $\lceil 3.8 \rceil = 4$
  - $\lfloor -3.8 \rfloor = -4$
  - $\lceil -3.8 \rceil = -3$
- Note that, for negative numbers, this is not the same as truncation.

Rounding

Definition (The Round Function)

Let $x \in \mathbb{R}$. Define $\langle x \rangle = \lfloor x + 1/2 \rfloor$.

- We see that $\langle x \rangle$ is the value of x, rounded to the nearest integer.
- If x is exactly halfway between two integers, then it is rounded *up* to the next largest integer.

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Theorem

Theorem

For all $x \in \mathbb{R}$ *and all* $n \in \mathbb{Z}$, $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.

Theorem

Proof.

- Let $x \in \mathbb{R}$ and let $n \in \mathbb{Z}$.
- Let $m = \lfloor x \rfloor \in \mathbb{Z}$ and let $e = x - m$.
- Then $x = m + e$ and $0 \leq e < 1$.
- Then

$$\begin{aligned} \lfloor x + n \rfloor &= \lfloor (m + e) + n \rfloor \\ &= \lfloor (m + n) + e \rfloor \\ &= m + n \\ &= \lfloor x \rfloor + n. \end{aligned}$$

□

Conjectures

- Is there a comparable statement involving $\lceil x + n \rceil$?
- Is it true that $\forall x, y \in \mathbb{R}$, $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$?
- Is it true that $\forall x \in \mathbb{R}$, $\lfloor 2x \rfloor = 2\lfloor x \rfloor$?
- Is it true that $\forall x \in \mathbb{R}$, $\lfloor x^2 \rfloor = (\lfloor x \rfloor)^2$?
- Is it true that $\forall x \in \mathbb{R}$, $\lfloor x + 1/2 \rfloor = \lceil x - 1/2 \rceil$?
- If they are not true for all $x, y \in \mathbb{R}$, then are they true for some $x, y \in \mathbb{R}$? Which ones?

Theorem

Theorem

For all real numbers x, $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor 2x \rfloor$.

Theorem

Proof.

- Let $x \in \mathbb{R}$.
- Let $n = \lfloor x \rfloor$ and $e = x - n$.
- Either $0 \leq e < \frac{1}{2}$ or $\frac{1}{2} \leq e < 1$.

□

Proof

Proof.

Case 1: Suppose $0 \le e < 1/2$.

Then

$$1/2 \le e + 1/2 < 1.$$

Therefore

$$\begin{aligned} \lfloor x \rfloor + \lfloor x + 1/2 \rfloor &= n + \lfloor n + e + 1/2 \rfloor \\ &= n + n \\ &= 2n \\ &= 2\lfloor x \rfloor. \end{aligned}$$

□

Proof

Proof.

Case 2: Suppose $1/2 \leq e < 1$.

Then,

$$0 \leq e - 1/2 < 1/2.$$

Therefore

$$\begin{aligned}
\lfloor x \rfloor + \lfloor x + 1/2 \rfloor &= n + \lfloor n + e + 1/2 \rfloor \\
&= n + \lfloor (n+1) + (e - 1/2) \rfloor \\
&= n + (n+1) \\
&= 2n + 1.
\end{aligned}$$

□

Proof

Proof.

Also, $0 \leq 2e - 1 < 1$.

Therefore,

$$\begin{aligned} \lfloor 2x \rfloor &= \lfloor 2n + 2e \rfloor \\ &= \lfloor 2n + 1 + (2e - 1) \rfloor \\ &= 2n + 1, \end{aligned}$$

Therefore $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = \lfloor 2x \rfloor$.

□

Proof

Proof.

- Therefore, for all $x \in \mathbb{R}$, $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = \lfloor 2x \rfloor$.

$\square$

- Write a similar statement using the ceiling function.
- Write a similar statement using the round function.

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Binary Search

- In a binary search of a list of size n, we begin by comparing the value to the middle element.
- If it matches, we are done. If it fails to match, we continue searching in the same manner in the lower half or the upper half.
- How many comparisons are required to find the value we are looking for?

Binary Search

- Suppose $n = 225$ and the elements are `a[0]` through `a[224]`.
- We first compare to element `a[112]`.
- Next, we compare to element `a[65]` or `a[168]`, and so on.

Binary Search

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- We eliminate 1 more and divide: 27 or 28 in the halves.
- Again: 13 or 14.

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- Again: 6 or 7.

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- Again: 13 or 14.
- Again: 6 or 7.
- Again: 2 or 3.

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- Again: 0 or 1.
- Done.

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- We eliminate 1 and divide the rest in half: 112 in each half.
- We eliminate 1 more and divide the rest in half: either 55 or 56 in the halves.
- We eliminate 1 more and divide: 27 or 28 in the halves.
- Again: 13 or 14.
- Again: 6 or 7.
- Again: 2 or 3.
- Again: 0 or 1.
- Done.

Binary Search

- Consider the calculation:

$$
\begin{aligned}
\lfloor 225 \div 2 \rfloor &= 112, \\
\lfloor 112 \div 2 \rfloor &= 56, \\
\lfloor 56 \div 2 \rfloor &= 28, \\
\lfloor 28 \div 2 \rfloor &= 14, \\
\lfloor 14 \div 2 \rfloor &= 7, \\
\lfloor 7 \div 2 \rfloor &= 3, \\
\lfloor 3 \div 2 \rfloor &= 1.
\end{aligned}
$$

Binary Search

- The maximum number of comparisons is $\lceil \log_2 225 \rceil = 8$.
- The minimum number of comparisons is $\lfloor \log_2 225 \rfloor = 7$.
- In general, the number is either $\lfloor \log_2 n \rfloor$ or $\lceil \log_2 n \rceil$.

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- How many comparisons are required when $n = 10000$?

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- The minimum number of comparisons is $\lfloor \log_2 225 \rfloor = 7$.
- In general, the number is either $\lfloor \log_2 n \rfloor$ or $\lceil \log_2 n \rceil$.
- How many comparisons are required when $n = 10000$?
- When $n = 100000$?

Binary Search

- The maximum number of comparisons is $\lceil \log_2 225 \rceil = 8$.
- The minimum number of comparisons is $\lfloor \log_2 225 \rfloor = 7$.
- In general, the number is either $\lfloor \log_2 n \rfloor$ or $\lceil \log_2 n \rceil$.
- How many comparisons are required when $n = 10000$?
- When $n = 100000$?
- When $n = 10^{12}$?

Puzzle

- How many trailing zeros are there in the decimal representation of 1000!?
- $(1000! = 1000 \cdot 999 \cdot 998 \cdots 2 \cdot 1.)$

Puzzle

- A trailing zero is produced by, and only by, a factor of 10.
- A factor of 10 is produced by, and only by, a pair of prime factors 2 and 5.
- How many pairs of factors 2 and 5 are there in 1000!?

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- Read Section 4.5, pages 191 - 196.
- Exercises 4, 8, 9, 10, 15, 16, 20, 21, 24, 25, page 197.