Test of Goodness of Fit

Lecture 48
Sections 14.1 - 14.3

Robb T. Koether
Hampden-Sydney College
Mon, Apr 23, 2012
Outline

1. Introduction
2. The Goodness-of-Fit Test
3. The Chi-Square Statistic
4. The Chi-Square Distribution
5. Goodness-of-Fit Test on the TI-83
6. One Last Comment
7. Assignment
In Chapter 9, we learned how to compare a proportion $p$ to a hypothetical value $p_0$. For example, we tested a coin to see whether it was fair. More specifically, we tested to see whether the proportion of heads was 0.50. We could just as easily have tested whether the proportion of tails was 0.50. But what do we do if we have more than two categories? For example, Republican, Democrat, Independent. Christian, Muslim, Jewish, Atheist. Strongly approve, Approve, Disapprove, Strongly disapprove, No opinion.
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Example (Goodness-of-Fit Test)

- I would like to test whether a die is fair.
- I rolled the die 90 times and obtained the following results:

<table>
<thead>
<tr>
<th>Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>9</td>
<td>15</td>
<td>19</td>
<td>14</td>
<td>18</td>
<td>15</td>
</tr>
</tbody>
</table>

Does the die appear to be fair?
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<td>9</td>
<td>15</td>
<td>19</td>
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<td>18</td>
<td>15</td>
</tr>
</tbody>
</table>

Does the die appear to be fair?
To answer this question, we must return to the same question that we have asked many times already.

Is the difference between what we observed and what we would expect to observe from a fair die small enough that we can reasonably attribute it to randomness?
The Hypotheses

- The null hypothesis specifies the probability (or proportion) for each category.
- The alternative hypothesis simply states that $H_0$ is false.
The Hypotheses

Example (Steps 1 and 2)

(1) $H_0 : p_1 = 1/6, p_2 = 1/6, p_3 = 1/6, p_4 = 1/6, p_5 = 1/6, p_6 = 1/6.$

$H_1 : H_0$ is false.

(2) $\alpha = 0.05.$
Definition (Observed and Expected Counts)

The **observed counts** are the counts that were actually observed in the sample. The **expected counts** are the counts that one would expect to observe if the null hypothesis were true.
Expected Counts

- To find the expected counts, we apply the hypothetical proportions to the sample size.
- For example, the hypothetical proportion for rolling a 1 is $1/6$, so we compute $1/6$ of 90:
  \[
  \frac{1}{6} \times 90 = 15.
  \]
- Do *not* round the values off to whole numbers.
The Test Statistic

- Make a chart showing both the observed counts and the expected counts (in parentheses).

<table>
<thead>
<tr>
<th>Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed</td>
<td>9</td>
<td>15</td>
<td>19</td>
<td>14</td>
<td>18</td>
<td>15</td>
</tr>
<tr>
<td>(Expected)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
</tr>
</tbody>
</table>

- The expected counts are based on the hypothetical proportions.
The Test Statistic

- Denote the observed counts by $O$ and the expected counts by $E$.
- Define the chi-square ($\chi^2$) statistic to be

$$\chi^2 = \sum_{\text{all cells}} \frac{(O - E)^2}{E}.$$
Example (Step 3)

(3) \[ \chi^2 = \sum_{\text{all cells}} \frac{(O - E)^2}{E}. \]
The Value of the Test Statistic

- Clearly, if *all* of the deviations $O - E$ are small, then $\chi^2$ will be small.
- But if *even a few* of the deviations $O - E$ are large, then $\chi^2$ will be large.
The Value of the Test Statistic

Example (Step 4)

<table>
<thead>
<tr>
<th>Number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed</td>
<td>9</td>
<td>15</td>
<td>19</td>
<td>14</td>
<td>18</td>
<td>15</td>
</tr>
<tr>
<td>(Expected)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
<td>(15)</td>
</tr>
</tbody>
</table>

\[
\chi^2 = \frac{(9 - 15)^2}{15} + \frac{(15 - 15)^2}{15} + \frac{(19 - 15)^2}{15} + \frac{(14 - 15)^2}{15} + \frac{(18 - 15)^2}{15} + \frac{(15 - 15)^2}{15} \\
= \frac{36}{15} + \frac{0}{15} + \frac{16}{15} + \frac{1}{15} + \frac{9}{15} + \frac{0}{15} \\
= \frac{62}{15} \\
= 4.133.
\]
Compute the $p$-Value

- The $p$-value for this example is the likelihood of observing a $\chi^2$ value as large at 4.133, if the die is fair.
- To find that value, we need to know something about the distribution of $\chi^2$ for a fair die.
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Chi-Square Degrees of Freedom

**Definition (χ² degrees of freedom)**

In a goodness-of-fit test, the number of degrees of freedom is one less than the number of cells.

- The χ² distribution has an associated degrees of freedom, just like the t distribution.
- Each χ² distribution has a slightly different shape, depending on the number of degrees of freedom.
- For example, we let χ⁵ denote the chi-square statistic with 5 degrees of freedom.
The Graph of $\chi^2_1$. 
The Graph of $\chi^2_2$. 
Chi-Square Degrees of Freedom

The Graph of $\chi^2_3$. 
The Graph of $\chi^2_4$. 
Chi-Square Degrees of Freedom

The Graph of $\chi^2_5$. 

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Chi-Square Degrees of Freedom

The Graph of $\chi^2_6$. 
The Graph of $\chi^2_7$. 

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The Graph of $\chi^2_8$. 

![Graph of $\chi^2_8$.]
The Graph of $\chi^2_9$. 
The Graph of $\chi^2_{10}$. 
The Graphs of $\chi_1^2, \chi_2^2, \ldots, \chi_{10}^2$. 

Degrees of Freedom
Properties of $\chi^2$

- The chi-square distribution with $df$ degrees of freedom has the following properties.
  - $\chi^2 \geq 0$.
  - It is unimodal.
  - It is skewed right (not symmetric!)
  - $\mu_{\chi^2} = df$.
  - $\sigma_{\chi^2} = \sqrt{2df}$.
  - If $df$ is large, then $\chi^2_{df}$ is approximately normal with mean $df$ and standard deviation $\sqrt{2df}$. 

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TI-83 - Chi-Square Probabilities

**TI-83 Chi-square Probabilities**

- Press **2nd DISTR**.
- Select **χ²cdf**.
- Enter the lower endpoint, the upper endpoint, and the degrees of freedom.
- Press **ENTER**. The probability appears in the display.
Practice

- Find $P(\chi^2_3 > 6)$.
- Find $P(4 < \chi^2_8 < 12)$.
- Find $P(\chi^2_5 < 10)$.
- Find $P(\chi^2_5 > 4.133)$. 
In our example, we found $\chi^2 = 4.133$.
There are 6 categories (1 - 6), so there are 5 degrees of freedom.
Example (Steps 5, 6, and 7)

(5) \( p\)-value = \( \chi^2 \text{cdf}(4.133, 0.99, 5) = 0.5304 \).

(6) Accept \( H_0 \).

(7) The die is fair.
Be careful when using the TI-83!

There is a function called $\chi^2$-Test, but it does not perform the goodness-of-fit test.

Some TI-84s have a GOF-Test function.

The GOF-Test function does perform the goodness-of-fit test.
Goodness-of-Fit Test on the TI-83

TI-83 Goodness-of-fit test

- Put the observed counts in list $L_1$.
- Put the hypothetical proportions in list $L_2$.
- Multiply $L_2$ by the sample size and store as $L_3$. These are the expected counts.
- Calculate $\left( L_1 - L_3 \right)^2 / L_3$ (either all at once or step by step).
- Go to LIST > MATH and select sum (item #5).
- Enter Ans and press ENTER. The value of $\chi^2$ appears.
- Then use $\chi^2 cdf$ to find the $p$-value.
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One Last Comment

- It so happens that the die used in that example was not fair.
- The probability of each of 1, 2, 3, and 4 was 15% and the probability of each of 5 and 6 was 20%.
- Why did the test show that it was fair?
Assignment

Homework

- Exercises 1 - 5, page 928.