1. (a) A counterexample is sufficient. That is, find $x_1$ and $x_2$, not equal, such that $f(x_1) = f(x_2)$. The simplest example is $x_1 = 0$ and $x_2 = 2$.

(b) Suppose that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{R} - \{0\}$. Then

\[
\frac{x_1 - 1}{x_1} = \frac{x_2 - 1}{x_2},
\]

\[
x_2(x_1 - 1) = x_1(x_2 - 1),
\]

\[
x_2x_1 - x_2 = x_1x_2 - x_1
\]

\[
-x_2 = -x_1
\]

\[
x_2 = x_1
\]

Therefore, $f$ is one-to-one.

2. (a) Yes, $f$ is a function. For every element $A \in \mathcal{P}(X)$, $f(A)$ is a uniquely defined element in $\mathcal{P}(X)$.

(b) $f(\{a, c\}) = \{a, c\} \cup \{a, b\} = \{a, b, c\}$.

(c) No, $f$ is not one-to-one. For example, $f(\{a, c\}) = f(\{c\})$.

(d) No, $f$ is not onto. Since $f(A) = A \cup \{a, b\}$, clearly $\{a, b\} \subset f(A)$ for all $A \in \mathcal{P}(X)$. Therefore, $f(A) \neq \emptyset$ for any $A \in \mathcal{P}(X)$.

3. Group the strings first by length. Let $S_n$ be the set of all strings of length $n$, for all $n \geq 0$. Within each group, arrange the strings in alphabetical order. Then list the elements of $S$ by listing the elements of $S_0, S_1, S_2, \ldots$ in that order.

4. First, show that the statement is true in the base cases when $n = 0$ and $n = 1$. In those cases, $a_0 = 0$ and $F_0 - 1 = 1 - 1 = 0$, and $a_1 = 0$ and $F_1 - 1 = 1 - 1 = 0$. So $a_n = F_n - 1$ when $n = 0$ or 1.

Now prove the general case. Suppose that the statement is true for all $k$, such that $1 \leq k \leq n$, for some $n \geq 2$. Then

\[
a_{n+1} = a_n + a_{n-1} + 1
\]

\[
= (F_n - 1) + (F_{n-1} - 1) + 1
\]

\[
= (F_n + F_{n-1}) - 1
\]

\[
= F_{n+1} - 1
\]

Therefore, the statement is true for all $n \geq 0$.

5. You may begin by computing the first several terms.

\[
b_0 = 0
\]

\[
b_1 = 3 \cdot 0 + 2 = 2
\]

\[
b_2 = 3 \cdot 2 + 2 = 8
\]

\[
b_3 = 3 \cdot 8 + 2 = 26
\]

\[
b_4 = 3 \cdot 26 + 2 = 80
\]

\[
b_5 = 3 \cdot 80 + 2 = 242
\]
At this point you might recognize these numbers as each being 1 less than a power of 3. Specifically, \( b_n = 3^n - 1 \).

Or you may use the method that was presented in class. The formula we are looking for must be of the form \( n_n = A \cdot 3^n + B \), for some real numbers \( A \) and \( B \). Substitute \( n = 0 \) and \( n = 1 \) to get two equations in \( A \) and \( B \):

\[
\begin{align*}
b_0 &= A \cdot 3^0 + B \\
b_1 &= A \cdot 3^1 + B
\end{align*}
\]

which is the system of equations

\[
\begin{align*}
0 &= A + B \\
2 &= 3A + B
\end{align*}
\]

Now solve the system of equations for \( A \) and \( B \). We get \( A = 1 \) and \( B = -1 \). So the solution is \( b_n = 3^n - 1 \).

6. If \( f(x) \) is \( \Omega(g(x)) \), then there exist positive real numbers \( M_1 \) and \( x_1 \) such that

\[
|f(x)| \geq M_1 |g(x)|,
\]

for all \( x > x_1 \). And if \( g(x) \) is \( \Omega(h(x)) \), then there exist positive real numbers \( M_2 \) and \( x_2 \) such that

\[
|g(x)| \geq M_2 |h(x)|,
\]

for all \( x > x_2 \). Then it follows that

\[
|f(x)| \geq M_1 |g(x)| \geq M_1 (M_2 |h(x)|) = (M_1M_2) |h(x)|,
\]

for all \( x > \text{max}(x_1, x_2) \). Thus, \( f(x) \) is \( \Omega(h(x)) \).

7. We must show that \( |x^3 + 5x^2 - 10x + 4| \leq M |x^3| \), for some \( M \in \mathbb{R} \) and for all \( x > x_0 \) for some \( x_0 \in \mathbb{R} \). For all \( x > 1 \),

\[
\begin{align*}
|x^3 + 5x^2 - 10x + 4| &\leq |x^3| + |5x^2| + |10x| + |4| \\
&= |x^3| + 5|x^2| + 10|x| + 4 \\
&< |x^3| + 5|x^3| + 10|x^3| + 4|x^3| \\
&= 20|x^3|
\end{align*}
\]

Therefore, \( x^3 + 5x^2 - 10x + 4 \) is \( O(x^3) \).

8. (a) We are given \( x_0 = 1000 \) and \( t_0 = 10 \) \( \mu s \). The general formula for the runtime is

\[
t = t_0 \left( \frac{x \log x}{x_0 \log x_0} \right) = 10 \left( \frac{x \log x}{1000 \log 1000} \right) = \frac{x \log x}{300} \mu s.
\]

(b) If \( x = 10^6 \), then \( t \) is

\[
t = \frac{10^6 \log 10^6}{300} = 20000 \mu s = 20 \text{ ms} = 0.02 \text{ sec}.
\]