1. (12 pts)
   (a) Let \( f : \mathbb{R} \to \mathbb{R}, \) defined by \( f(x) = x(x-2) \). Prove that \( f \) is not one-to-one.
   (b) Let \( f : \mathbb{R} - \{0\} \to \mathbb{R}, \) defined by \( f(x) = \frac{x-1}{x} \). Prove that \( f \) is one-to-one.

2. (16 pts) Let \( X = \{a, b, c, d\} \) and define a function \( f : \mathcal{P}(X) \to \mathcal{P}(X) \) by \( f(A) = A \cup \{a, b\}. \) (\( \mathcal{P}(X) \) is the power set of \( X \).)
   (a) Is \( f \) a function? (Yes or no)
   (b) Find \( f(\{a, c\}) \).
   (c) Is \( f \) one-to-one? (Yes or no)
   (d) Is \( f \) onto? (Yes or no)

3. (12 pts) Let \( S \) be the set of all finite strings consisting of the letters \( a \) and \( b \). Demonstrate that \( S \) is countably infinite by describing a method of listing the elements of \( S \) in such a way that each string in \( S \) occurs exactly once in the list.

4. (12 pts) The Fibonacci numbers \( F_n \) may be computed recursively by the formulas

\[
\begin{align*}
F_1 &= 1 \\
F_2 &= 1 \\
F_n &= F_{n-1} + F_{n-2} \text{ for all } n \geq 3.
\end{align*}
\]

Let \( a_n \) be the number of additions required to compute \( F_n \) recursively. Clearly,
\[
\begin{align*}
a_1 &= 0 \\
a_2 &= 0 \\
a_n &= a_{n-1} + a_{n-2} + 1 \text{ for all } n \geq 3.
\end{align*}
\]

Use induction to show that \( a_n = F_n - 1 \) for all \( n \geq 1 \).

5. (12 pts) Let \( b_n \) be defined recursively as

\[
\begin{align*}
b_0 &= 0 \\
b_n &= 3b_{n-1} + 2 \text{ for all } n \geq 1.
\end{align*}
\]

Find a non-recursive formula for \( b_n \). You need not prove that the formula is correct.

6. (12 pts) Let \( f, g, \) and \( h \) be functions from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \). Use the definition of the \( \Omega \)-notation to prove that if \( f(x) \) is \( \Omega(g(x)) \) and \( g(x) \) is \( \Omega(h(x)) \), then \( f(x) \) is \( \Omega(h(x)) \).
7. (12 pts) Prove from the definition of the $O$-notation that $x^3 + 5x^2 - 10x + 4$ is $O(x^3)$.

8. (12 pts) Suppose that a program has a run time that is $\Theta(x \log x)$ and that when the input size is 1000, the run time is 10 ms.

   (a) Find a formula for the runtime as a function of the input size.
   (b) Estimate the run time when the input size is $10^6$. 