Math and Society — Math 111 Midterm Exam I

Name: Solution Guide

Instructions: You must show all work to earn full credit. If a problem asks for an explanation, you must answer with a complete sentence. If you do not have room in the given space to answer a question, use the back of another sheet and *indicate clearly* which work goes with which problem.

Problem	Maximum Points	Your Points
1	10	
2	20	
3	15	
4	20	
5	20	
6	15	
Total	100	

1. (10 points) Suppose that Alice and Bob enter a contest together and win a Winnebago. Alice values the Winnebago at $90,000 while Bob thinks that it is worth $110,000. They agree to use the method of sealed bids to decide who should get the Winnebago.

   (a) What are the fair dollar shares (FDS) for each player?

   Alice $45,000
   Bob $55,000

   (b) Who will get the Winnebago?

   Bob thinks that it is worth more, so he gets it.

   (c) Will one player owe the other player any money? If so, who and how much?

   Yes, Bob owes the "estate" his FDS $55,000. Of this, $45,000 should go to Alice since that is her FDS.

   (d) Will there be a surplus? How should it be divided?

   Yes, there is a surplus of $\$55,000 - \$45,000 = \$10,000$ which should be divided evenly, $5,000 to Bob and $5,000 to Alice.

2. (20 points) Use the following preference schedule to answer the questions below.

Ice Cream Party Preferences			
Number of voters	**8**	**7**	**2**
1st choice	Chocolate	Vanilla	Strawberry
2nd choice	Vanilla	Strawberry	Vanilla
3rd choice	Strawberry	Chocolate	Chocolate

(a) How many people prefer vanilla as their second choice?

10 people.

(b) In a head to head competition between chocolate and vanilla, which flavor would get more votes?
8 People prefer chocolate over vanilla but 9 people prefer vanilla more than chocolate, so vanilla would win.

(c) Is there a Condorcet candidate? Explain.

Yes, vanilla wins against both of the other two candidates in head-to-head match-ups, thus vanilla is the Condorcet candidate.

(d) If the Borda count method were used to determine the winning flavor, how many points would each flavor get?

Chocolate gets $8 \times 3 + 7 \times 1 + 2 \times 1 = 33$ points
Vanilla gets $8 \times 2 + 7 \times 3 + 2 \times 2 = 41$ points
Strawberry gets $8 \times 1 + 7 \times 2 + 2 \times 3 = 28$ points

3. (15 points) A cake worth \$18 is being divided by three people. Suppose they agree to use the lone-chooser method, so that there is one chooser C and two dividers D_1 and D_2.

   (a) What is a fair share of the cake worth for each player?

   A fair share is $\frac{1}{3}$ of the value of the cake, so $\frac{1}{3}(18) = \$6$.

   (b) Using the lone-chooser method, how many pieces of cake will there be at the end?

   There will be 6 pieces of cake.

   (c) When D_1 and D_2 first cut the cake, they use the divider-chooser method to cut it in half. Suppose that D_1 is the divider. How much will each half be worth in D_1 eyes?

   D_1 will be careful to cut the cake into two equal pieces since he doesn't know which half he will end up with. Therefore each half is worth \$9 in his eyes.

   (d) Suppose that D_1 cut the cake into halves that were worth \$6 and \$12 in D_2's opinion. At the very end, what is the value of the cake that D_2 gets (in D_2's opinion)?

   Since D_2 gets to pick the half, she will pick the half worth more (\$12). She then cuts this half into three equal pieces worth \$4 each (in her eyes). After the chooser takes one of those pieces, she will have \$8 worth of cake.

   (e) Lone-chooser is one of the three methods that can divide a cake fairly between N people. What are the names of the other two methods that we covered?

   Last Diminisher and Lone-Divider.

4. (20 points) Use the weighted voting system $[15 : 8, 5, 4, 3]$ to answer the following questions.

(a) What is the quota? 15

(b) How many players are there? 4

(c) What is the weight of player P_2? 5

(d) Is there a dictator? No

(e) How many winning coalitions contain exactly three players? What are they?

There are three winning coalitions:

$$\{P_1, P_2, P_3\} \quad \{P_1, P_2, P_4\} \quad \{P_1, P_3, P_4\}$$

(f) Write down all of the winning coalitions (not just 3 player ones) and indicate the critical players with a star (*).

$$\{P_1^*, P_2^*, P_3^*\} \quad \{P_1^*, P_2^*, P_4^*\} \quad \{P_1^*, P_3^*, P_4^*\} \quad \{P_1^*, P_2, P_3, P_4\}$$

Everyone is critical in the 3-player coalitions since there are no winning 2-player coalitions. Only P_1 is indispensible in the grand coalition, so P_1 is the only critical player there.

(g) Calculate the Banzhaf power index of each player.

The number of times that anyone is critical is $T = 3 + 3 + 3 + 1 = 10$. Player P_1 is critical $B_1 = 4$ times, and players P_2, P_3, and P_4 are each critical two times. Thus

$$\begin{aligned} \beta_1 &= 4/10 = 40\% \\ \beta_2 &= 2/10 = 20\% \\ \beta_3 &= 2/10 = 20\% \\ \beta_4 &= 2/10 = 20\% \end{aligned}$$

(h) If we wanted to calculate the Shapley-Shubik power index, how many sequential coalitions would there be?

There are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ sequential coalitions.

5. (20 points) A small town has four bus routes. The average number of passengers for each bus route is listed below. The town has 20 buses and would like to apportion the buses so that each route gets a number of buses proportional to the average number of riders.

Bus Routes				
Route	**A**	**B**	**C**	**D**
Average # of riders	432	272	831	465

(a) In this apportionment problem, what are the "seats" and what are the "states"?

The seats are the buses and the states are the bus routes.

(b) What is the total population?

The total population is $P = 432 + 272 + 831 + 465 = 2000$.

(c) What is the standard divisor?

The total standard divisor is $SD = \dfrac{P}{M} = \dfrac{2000}{20} = 100$.

(d) Find the standard quota for each route.

Route	A	B	C	D
Standard Quota	4.32	2.72	8.31	4.65

(e) Using Hamilton's method, how many buses would each route get?

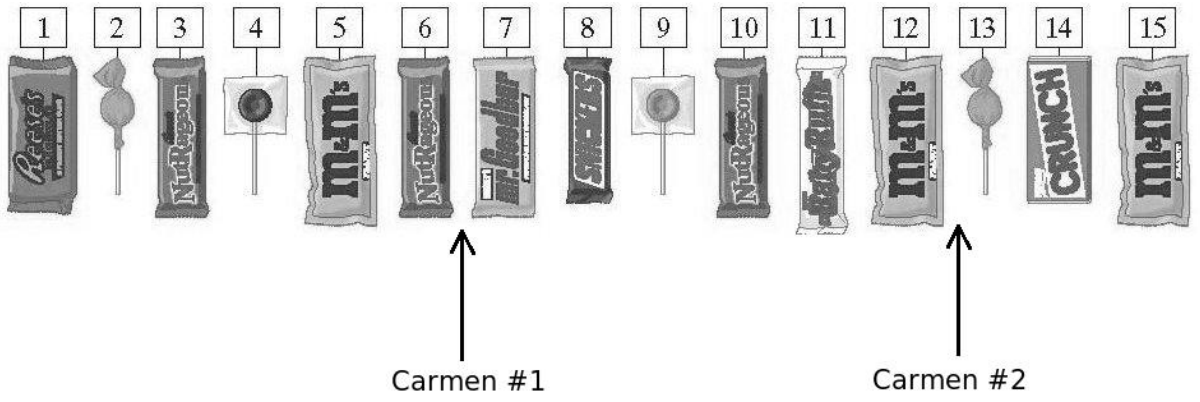
Using Hamilton's method each route is apportioned its lower quota in buses. That uses up $4 + 2 + 8 + 4 = 18$ buses, leaving 2 extras. The extra buses are given to routes B and D because they have the highest decimal places So the buses are assigned as follows, route A: 4, route B: 3, route C: 8, and route D: 5.

(f) **(Extra Credit)** What is the apportionment using Webster's method?

In Webster's method, the quotas are rounded to the nearest integer and a suitable divisor is choosen so that the rounded quotas add up to M. The rounded quotas are $A = 4$, $B = 3$, $C = 8$, and $D = 5$, which add up to 20, so that is the answer.

6. (15 points) Alice, Bob, and Carmen use the method of markers to divide 15 pieces of candy. Suppose that

   - Alice likes everything equally.
   - For Bob, each piece of candy is worth 25¢ except the Reese's peanut butter cups which are worth $1.



(a) How much is all of the candy worth to Bob? How much would a fair share be worth?

All of the candy is worth $14 \times \$0.25 + \$1.00 = \$4.50$. Bob's fair share would be $\frac{1}{3}$ of \$4.50 or \$1.50.

(b) Carmen has already placed markers dividing the candy into shares that she would agree to. Clearly indicate where Alice and Bob will place their markers using the picture above.

Alice will place markers immediately after #5 and again after #10.
Bob will place markers after #3 and after #9.

(c) Who gets the first share? Who gets the second share? And who gets the third share?

Bob gets his first share, Alice gets her second share and Carmen gets her third share.

(d) Is there any leftover candy? Which pieces are left?

Yes, the pieces remaining are #4, 5 and #11, 12.