Algebraic Structures Homework #1
Due Friday, September 13

Write up detailed solutions for each of the problems below. Remember, only half of your grade will be based on mathematical accuracy. The other half will depend on the clarity and neatness of your exposition.

(1) Recall that $a \equiv b \pmod{n}$ (that is, "equivalent modulo n") if and only if $a - b$ is divisible by n. Prove that this relation is an equivalence relation on the integers by showing that it is transitive (we already showed that it is reflexive and symmetric in class).

(2) Prove that if $a \equiv a' \pmod{n}$ and $b \equiv b' \pmod{n}$, then $ab \equiv a'b' \pmod{n}$.

(3) Let n be an integer. Prove that if n is not divisible by 3, then n^2 is not divisible by 3.

(4) For every positive integer n, use induction to prove that a set with exactly n elements has exactly 2^n subsets (counting the empty set and the entire set).

(5) Use induction to prove that the Towers of Hanoi puzzle with n disks can always be solved (Extra credit if you show that it can always be solved in $2^n - 1$ steps).

Homework Exposition Guidlines

In order to receive full credit for the exposition part of the grade, your homework must satisfy the following requirements.

- It must be neat, legible, stapled (if more than one sheet of paper), and have your name written clearly on top.
- Individual problems must be clearly labeled and separated from other problems by at least a full line.
- Each exercise must include a description of the problem to be solved (this may be copied directly from the exercise itself).
- All explanations must be written in complete sentences.
- The end of any proof should be indicated with a Q.E.D. or a small square.

Example Problem Write-Ups

1. Claim: The sum of any two even integers is even.

Proof. Suppose a and b are even integers. Since a is even, it is divisible by 2 and therefore there is an integer m such that $a = 2m$. Similarly, there is an integer n such that $b = 2n$. It follows that

$$a + b = 2m + 2n = 2(m + n).$$

Thus $a + b$ is divisible by 2 and so $a + b$ is even. □

2. Claim: $1 + 2 + ... + n = \dfrac{n(n+1)}{2}$.

Proof. We use induction to prove the claim for all positive integers n. Observe that the claim is true for $n = 1$. Now assume that the statement is true for some fixed k, i.e.,

$$1 + 2 + ... + k = \frac{k(k+1)}{2}.$$

Then,

$$1 + 2 + ... + k + (k+1) = \frac{k(k+1)}{2} + (k+1) =$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

This is precisely the claim for the case when $n = k + 1$. Therefore the claim is true for all positive integers n by the Principle of Induction. □