Math 441 - Mathematical Induction Workshop Solutions

Here are solutions to the workshop problems from yesterday. You can see how I wrote my solutions, but remember: my way is only one of many "correct" ways to write a proof!

1. Use induction to prove that a set with N elements has exactly 2^N subsets.

   Proof. Note that a set with one element has exactly two subsets, the set itself and the empty set. Using mathematical induction, let us suppose that every set with k elements has 2^k subsets. Now consider a set S with $k+1$ elements. Let us write $S = \{x_1, x_2, ..., x_k, x_{k+1}\}$. Note that any subset of S which does not contain x_{k+1} is contained in the set $\{x_1, x_2, ..., x_k\}$. Since $\{x_1, x_2, ..., x_k\}$ has k elements, there must be exactly 2^k subsets of S which do not contain x_{k+1}. Now let us count the subsets of S containing x_{k+1}. Each of these subsets must be a subset of $\{x_1, x_2, ..., x_k\}$ combined with the element x_{k+1}. Thus there are exactly 2^k subsets containing x_{k+1} for a total of $2^k + 2^k = 2^{k+1}$ subsets of S. Therefore every set with N elements has 2^N subsets by mathematical induction. □

2. Use induction to prove that the Towers of Hanoi problem with n discs can always be solved.

   Proof. The Towers of Hanoi puzzle is easy when $n = 1$. We may just move the single disk to the far peg. Suppose by induction that we know we can move k disks from one peg to another. If we have a tower of $k+1$ disks, then we begin by moving the top k disks to the middle peg. Then we can move the remaining largest disk to the farthest peg. Then we can move the k disks from the center peg over the to farthest peg. We have just shown by induction that the Towers of Hanoi puzzle can be solved for any number of disks. □

3. George Pólya suggested the following exercise: What is wrong with the following proof that all horses have the same color? *If there's only one horse, there's only one color. Suppose within any set of n horses, there is only one color. Now look at any set of* $n+1$ *horses. Number them:* $1, 2, 3, ..., n, n+1$. *Consider the sets* $\{1, 2, 3, ..., n\}$ *and* $\{2, 3, 4, ..., n+1\}$. *Each is a set of only n horses, therefore with each there is only one color. But the two sets overlap, so there must be only one color among all* $n+1$ *horses.*

   Hint. Does the proof work for $n = 2$? Why not?

4. The **Principle of Strong Induction** says that if $P(N)$ is logical statement about $N \in \mathbb{N}$ and if

   (a) $P(1)$ is true, and

   (b) If $P(k)$ is true for all $k \in \{1, 2, ..., N\}$, then $P(k+1)$ is true,
   Part (b) was incorrectly worded. It should say:
   If $P(k)$ is true for all $k \in \{1, 2, ..., N\}$, then $P(k)$ is true when $k = N + 1$,

   then $P(N)$ is true for all $N \in \mathbb{N}$.

   Use the Principle of Strong Induction to prove the **Fundamental Theorem of Arithmetic**, which states: "every integer $N \geq 2$ is a product of prime numbers."

   Proof. We first observe that the base case when $N = 2$ is trivial because 2 is already a product of prime numbers, namely the product of itself with nothing else. Let us suppose that we know that every integer from 2 up to N is a product of prime numbers. If we can prove that the same is true of $N + 1$, then we will be done by the Principle of Strong Induction. We will proceed using a proof by contradiction. Suppose that $N+1$ is not a product of prime numbers. Then $N+1$ cannot be a prime number itself, because it would be a trivial product of primes. So $N+1$ is a composite number... that is, there are two integers $a, b \geq 2$ such that $N + 1 = ab$. Furthermore, a and b must be smaller than $N + 1$. By the Principle of Strong Induction, both a and b must be products of prime numbers, so the combination ab is also a product of prime numbers. But this is a contradiction, since we assumed that $N + 1$ is not the product of primes. This contradiction proves that $N + 1$ is a product of primes and therefore by the Principle of Strong Induction all integers $N \geq 2$ are products of prime numbers. □

5. A group of n people play a round-robin tournament. Each game ends in either a win or a loss. Show that it is possible to label the players $P_1, P_2, P_3, ..., P_n$ in such a way that P_1 defeated P_2, P_2 defeated P_3, ... , P_{n-1} defeated P_n.

   This problem is harder then the others, but it is not impossible. It is worth 10 points of extra-credit towards any one homework assignment if you can solve it!