**Math 441 - Sequences Workshop**

1. We have used the following fact in class: *A sequence $s_n$ converges to $s$ if and only if every neighborhood of $s$ contains all but finitely many $s_n$.* In this workshop you will prove the statement.

   (a) Translate the phrase, *contains all but finitely many $s_n$*, into a precise mathematical statement using a variable $M$.

   It took us a while to figure out what this question was asking. Bascically, we have to restate the definition above into one using symbols. So for example, the phrase every neighborhood of $s$ becomes: $\forall N(s;\epsilon)$. In order to translate the phrase, "contains all but finitely many $s_n$", it may be helpful to switch to the equivalent phrase "contains an entire tail of the sequence".

   What is a tail of a sequence? It is a set that includes every $s_n$ past a certain point. Here are three different ways that you came up with to describe such a set:

   - $\{s_n : n \geq M\}$ for some $M \in \mathbb{N}$ ($\leftarrow$ Janko wrote the tail this way.)
   - $\{s_n : n \in \mathbb{N}\} \backslash \{s_1, s_2, ...s_M\}$ for some $M \in \mathbb{N}$ ($\leftarrow$ Matt suggested this way.)
   - $\{s_{n+M} : n \in \mathbb{N}\}$ for some $M \in \mathbb{N}$ ($\leftarrow$ Chris came up with this notation.)

   Each of these definitions of a tail are correct. Let's pick one, say the first one, to finish the translation:

   $$...contains\ all\ but\ finitely\ many\ s_n$$
   $$\shortparallel$$
   $$...contains\ a\ tail\ of\ the\ sequence\ s_n$$
   $$\shortparallel$$
   $$there\ exists\ a\ tail\ such\ that\ the\ tail\ is\ contained...$$
   $$\shortparallel$$
   $$\exists M \in \mathbb{N}\ such\ that\ \{s_n : n \geq M\} \subseteq ...$$

   (b) Now prove that the $\epsilon$-definition of a limit is equivalent to the one stated above.

   The new definition says in words that $s \rightarrow s_n$ iff every neighborhood of $s$ contains all but finitely many $s_n$. Using the translation above, we can re-write this as:

   $$s \rightarrow s_n \Leftrightarrow \forall N(s;\epsilon), \exists M \in \mathbb{N}\ such\ that\ \{s_n : n \geq M\} \subseteq N(s;\epsilon)$$

   Can you show that this is equivalent to the $\epsilon$-definition below:

   $$s \rightarrow s_n \Leftrightarrow \forall \epsilon > 0, \exists M \in \mathbb{N}\ such\ that\ |s - s_n| < \epsilon\ \forall n > M$$