Algebraic Structures **Homework #13 Solutions**

1. **Chapter 13, Problem 1.** Compute the G-equivalence classes (i.e, orbits) for the following group actions.

   (a) $G = GL_2(\mathbb{R})$ and $X = \mathbb{R}^2$.

   > **Solution:** The orbit of the zero vector is $\text{Orb}_G(0) = \{0\}$ since every matrix maps 0 to 0. The orbit of any other vector is all of $\mathbb{R}^2 \backslash \{0\}$, since there is an invertible matrix that will map any nonzero vector to any other.

   (b) $G = D_4$ and $X = \{1, 2, 3, 4\}$.

   > **Solution:** Since D_4 includes all four rotations, all of the orbits are the same:
   > $$\text{Orb}_G(1) = \text{Orb}_G(2) = \text{Orb}_G(3) = \text{Orb}_G(4) = \{1, 2, 3, 4\}.$$

   (c) $X = G$, where G acts by left multiplication.

   > **Solution:** Take any a, b in G, and notice that left multiplication by ba^{-1} maps a to b. So the group can map a to any b. That means that the orbit of a is everything. In other words, $\text{Orb}_G(a) = G$ for all $a \in G$.

   (d) $X = G$, where G acts by conjugation.

   > **Solution:** The orbits of a group acting on itself by conjugation are called conjugacy classes. That's all you need to say, although it might be nice to point out that the conjugacy class for the identity only contains the identity. Without knowing more about G, you can't say anything else about the other conjugacy classes.

   (e) Let $H \leq G$ and $\mathcal{L}_H$ be the set of left cosets of H. Then $X = \mathcal{L}_H$ where G acts on X by the action
   $$(g, xH) \mapsto gxH$$

   > **Solution:** As in part (c), we can find a $g \in G$ which will map x to any point in G. Therefore the orbit of any left coset is the whole set X.

2. **Chapter 13, Problem 4.** Let G be the additive group of real numbers. Let the action of $\theta \in G$ on the real plane $\mathbb{R}^2$ be given by rotating the plane counterclockwise about the origin through θ radians. Let P be a point on the plane other than the origin.

   (a) Show that $\mathbb{R}^2$ is a G-set.

   Solution: We HTS that:

   1. Rotating by θ_1 and then rotating by θ_2 is the same as rotating by $\theta_1 + \theta_2$.
   2. Rotating by the identity $0 \in \mathbb{R}$ leaves every vector fixed.

   Now that I written these two conditions clearly, it is obvious that they are true.

   (b) Describe geometrically the orbit containing P.

   Solution: If you rotate P through all possible angles, then you will get the circle centered at the origin that contains P.

   (c) Find the group G_P (i.e., $\mathrm{Stab}_G(P)$).

   Solution: The stabilizer subgroup of the origin is every rotation. If P is not the origin, then $\mathrm{Stab}_G(P)$ is the set of all rotation angles that don't actually change P. Any integer multiple of 2π works. So $\mathrm{Stab}_G(P) = 2\pi\mathbb{Z}$.

3. **Chapter 13, Problem 5.** Let $G = A_4$ and suppose G acts on itself by conjugation.

   (a) Determine the orbits of each element in G.

   Solution: It might help to have a list of all 12 elements in A_4. Here they are:

   $$\{(), (2,3,4), (2,4,3), (1,2)(3,4), (1,2,3), (1,2,4),$$

   $$(1,3,2), (1,3,4), (1,3)(2,4), (1,4,2), (1,4,3), (1,4)(2,3)\}$$

   Let's focus on the orbit of one 3-cycle: $(1,2,3)$. By theorem 6.9,

   $$\tau(1,2,3)\tau^{-1} = (\tau(1), \tau(2), \tau(3)), \text{ for all } \tau \in G.$$

   Therefore, we can find the orbit of $(1,2,3)$ by plugging each $\tau \in G$ into the formula above and seeing what we get.

   $$\mathrm{Orb}_G((1,2,3)) = \{(1,2,3), (2,4,3), (1,4,2), (1,3,4)\}$$

We can also find the orbit of $(1,3,2)$ using the same technique.

$$\mathrm{Orb}_G((1,3,2) = \{(1,3,2),(2,3,4),(1,2,4),(1,4,3)\}$$

Since we have found all eight 3-cycles in G, we don't need to find the orbits of any of the remaining 3-cycles. The orbit of the identity under conjugation is easy since $\tau()\tau^{-1} = ()$ for all τ.

$$\mathrm{Orb}_G(()) = \{()\}$$

For elements like $(1,2)(3,4) \in G$, we can use the fact that

$$\tau(1,2)(3,4)\tau^{-1} = \tau(1,2)\tau^{-1}\tau(3,4)\tau^{-1} = (\tau(1),\tau(2))(\tau(3),\tau(4))$$

to find the orbit.

$$\mathrm{Orb}_G((1,2)(3,4)) = \{(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\}$$

(b) Determine the isotropy (aka, stabilizer) subgroups for each element of G.

Solution: It helps to find the stabilizer subgroups if you use the orbit stabilizer theorem $|G| = |\mathrm{Orb}_G(g)||\mathrm{Stab}_G(g)|$. Since each 3-cycle has an orbit with 4 elements, and $|G| = 12$, it follows that the stabilizer subgroup of a 3-cycle in G has 3 elements. If σ is a 3-cycle, then σ stabilizes itself, and $()$ stabilizes σ, so

$$\mathrm{Stab}_G(\sigma) = \{(),\sigma,\sigma^2\}.$$

Any element of the form $(a,b)(c,d)$ has 3 elements in its orbit, so it must contain 4 elements in its stabilizer subgroup. Once you know that, it is not hard to verify that

$$\mathrm{Stab}_G((a,b)(c,d)) = \{(),(a,b)(c,d),(a,c)(b,d),(a,d)(b,c)\}.$$

Finally, the identity is easy, since everything stabilizes the identity, so

$$\mathrm{Stab}_G(()) = G.$$

4. **Chapter 13, Problem 20.** A group acts faithfully on a G-set X if the identity is the only element of G that leaves every element of X fixed. Show that G acts faithfully on X if and only if no two distinct elements of G have the same action on each element of x.

Solution: ($\Longrightarrow$) Suppose that G acts faithfully on X and yet $g_1 x = g_2 x$ for all $x \in X$. Then $g_1^{-1} g_2 x = x$ for all $x \in X$. Therefore $g_1^{-1} g_2 = e$ so $g_1 = g_2$.

($\Longleftarrow$) If $g_1 x = g_2 x$ for all $x \in X$ implies that $g_1 = g_2$, then $gx = x$ implies that $g = e$ immediately. Therefore G acts faithfully. $\blacksquare$

5. **Chapter 13, Problem 23.** Let $|G| = p^n$ and suppose that $|Z(G)| = p^{n-1}$ for p prime. Prove that G is abelian.

Solution: If $|Z(G)| = p^{n-1}$, then chose some $g \in G \backslash Z(G)$. Note that $Z(G) \subset \mathrm{Stab}_G(g)$. Also $g \in \mathrm{Stab}_G(g)$. Therefore $\mathrm{Stab}_G(g)$ is a subgroup of G containing at least $p^{n-1}+1$ elements. Since the only divisor of p^n greater than p^{n-1} is p^n, we use the theorem of Lagrange to conclude that $|\mathrm{Stab}_G(g)| = p^n$ and therefore $\mathrm{Stab}_G(g) = G$. Thus $hgh^{-1} = g$ for all $h \in G$. So $g \in Z(G)$, a contradiction. Therefore $|Z(G)|$ cannot equal p^{n-1} allowing us to conclude anything we want. $\blacksquare$