Algebraic Structures — Midterm 2 Review

Things You Should Know

- You should know the following definitions and concepts: *Direct product of groups, finitely generated group, normal subgroup, factor group, group homomorphism, kernel, ring, integral domain, field, unit, ring homomorphism, ideal, maximal ideal, prime ideal.*
- Here are some of the theorems that you should definitely know: *The Fundamental Theorem of Finitely Generated Abelian Groups, The Fundamental Theorem of Algebra, The Division Algorithm for Polynomials*
- You should know the following fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}(\sqrt{\alpha})$, $\mathbb{Z}_p$. You should also know which of the following rings are integral domains: $\mathbb{Z}, n\mathbb{Z}$, $\mathbb{Z}_n$, $M_2(\mathbb{R})$, $M_2(\mathbb{Z})$, $R[x]$ (when R is any commutative ring with identity), $F[x]$ (when F is any field).
- Make sure you review all old homework problems, particularly the short answer problems.

Example Problems

1. Consider the polynomial $p(x) = x^4 - x^3 - x - 1 \in \mathbb{R}[x]$.
   (a) Verify that $i \in \mathbb{C}$ is a root of $p(x)$.
   (b) Use the fact that complex roots of real polynomials occur in conjugate pairs to find another root of $p(x)$.
   (c) What are the two factors corresponding to the roots in parts (a) and (b)? What is the product of the two factors? Is it a real polynomial?
   (d) Use the division algorithm to factor $p(x)$.
   (e) What is the complete factorization of $p(x)$ over $\mathbb{C}$? What are all of the roots of $p(x)$?
   (f) Is $p(x)$ irreducible in $\mathbb{Q}[x]$? What about in $\mathbb{R}[x]$?
2. Up to isomorphism, list all possible abelian groups of order 24.
3. What is the multiplicative inverse of $1 + \sqrt{5}$ in the extension field $\mathbb{Q}(\sqrt{5})$?
4. Is $\mathbb{R}$ an ideal in $\mathbb{C}$? Why or why not?
5. Is the polynomial $x^2 - x - 1$ irreducible in $\mathbb{Z}_3$? If so, list the elements of the field $\mathbb{Z}_3[x]/\langle x^2 - x - 1\rangle$? If not, factor $x^2 - x - 1$.
6. Suppose that $N_1 \lhd G_1$ and $N_2 \lhd G_2$. Prove that $N_1 \times N_2 \lhd G_1 \times G_2$.
7. Prove that there are no nontrivial ideals in any field.
8. Prove that if G has only one subgroup of order 5, then that subgroup is normal.