Definition of an Ordered Field

A **field** is a set F that satisfies the following axioms.

Addition Axioms

1. F is **Closed under Addition.** If $x, y \in F$, there is an element $x + y \in F$ called the *sum* of x and y.
2. **Addition is Commutative**: $x + y = y + x$ for all $x, y \in F$.
3. **Addition is Associative**: $(x+y)+z = x+(y+z),\ \forall x, y, z \in F$.
4. **Additive Identity**: There is an element $0 \in F$ such that $x+0 = x$ for every $x \in F$.
5. **Additive Inverse**: For every $x \in F$, there is an element $-x \in F$ such that $x + (-x) = 0$.

Multiplication Axioms

1. F is **Closed under Multiplication.** If $x, y \in F$, there is an element $xy \in F$ called the *product* of x and y.
2. **Multiplication is Commutative**: $xy = yx$ for all $x, y \in F$.
3. **Multiplication is Associative**: $x(yz) = (xy)z,\ \forall x, y, z \in F$.
4. **Multiplicative Identity**: There is a unique element $1 \in F$ such that $1x = x$ for all $x \in F$.
5. **Multiplicative Inverses**: For every $x \in F\backslash\{0\}$, there is a unique element x^{-1} such that $xx^{-1} = 1$.
6. **Distributive Law**: $x(y + z) = xy + xz,\ \forall x, y, z \in F$.

An **ordered field** is a field F with a relation $<$ that satisfies the following four additional order axioms.

Order Axioms

1. **Trichotomy Law**: For all $x, y \in F$ exactly one of the following holds: $x = y$, $x < y$, or $y < x$.
2. $<$ **is Transitive**: $<$ is a transitive relation on F.
3. **Additive Property**: $x < y \Rightarrow x + z < y + z,\ \forall x, y, z \in F$.
4. **Multiplicative Property**: For all $x, y, z \in F$, if $x < y$ and $z > 0$, then $xz < yz$.