34th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 27, 2012

Fill out the individual registration form

1. Evaluate

$$\int_0^{\pi/2} \frac{\cos^4 x + \sin x\, \cos^3 x + \sin^2 x\, \cos^2 x + \sin^3 x\, \cos x}{\sin^4 x + \cos^4 x + 2\sin x\, \cos^3 x + 2\sin^2 x\, \cos^2 x + 2\sin^3 x\, \cos x}\, dx.$$

2. Solve in real numbers the equation $3x - x^3 = \sqrt{x+2}$.

3. Find nonzero complex numbers a, b, c, d, e such that

$$\begin{aligned} a+b+c+d+e &= -1 \\ a^2+b^2+c^2+d^2+e^2 &= 15 \\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e} &= -1 \\ \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+\frac{1}{e^2} &= 15 \\ abcde &= -1 \end{aligned}$$

4. Define $f(n)$ for n a positive integer by $f(1) = 3$ and $f(n+1) = 3^{f(n)}$. What are the last two digits of $f(2012)$?

5. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n} - \left(\frac{1}{\ln n}\right)^{(n+1)/n}$ is convergent.

6. Define a sequence (a_n) for n a positive integer inductively by $a_1 = 1$ and $a_n = \dfrac{n}{\prod_{\substack{d|n \\ 1\le d<n}} a_d}$. Thus $a_2 = 2$, $a_3 = 3$, $a_4 = 2$ etc. Find a_{999000}.

7. Let A_1, A_2, A_3 be 2×2 matrices with entries in $\mathbb{C}$ (the complex numbers). Let tr denote the trace of a matrix (so $\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+d$). Suppose $\{A_1, A_2, A_3\}$ is closed under matrix multiplication (i.e. given i, j, there exists k such that $A_iA_j = A_k$), and $\operatorname{tr}(A_1+A_2+A_3) \neq 3$. Prove that there exists i such that $A_iA_j = A_jA_i$ for all j (here i, j are 1, 2 or 3).