1. (6 points) Find a unit vector that points in the same direction as the vector $\mathbf{a} = (3, 0, 4)$.

Solution: A unit vector has length one, so divide $\mathbf{a}$ by its length:

$$\mathbf{a}/||\mathbf{a}|| = (3/5, 0, 4/5).$$

2. (6 points) Find the vector equation for the line passing through $P = (3, -2, 7)$ and $Q = (2, -2, 5)$.

Solution: Use $P - Q$ as the direction vector. $P - Q = (1, 0, 2)$. Use either point as the initial point. One possible solution is:

$$(3, -2, 7) + t(1, 0, 2).$$

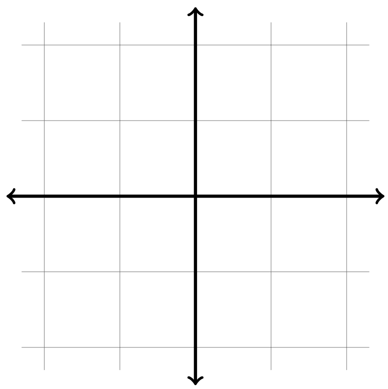
3. (8 points) Find the area of the parallelogram with vertices $A = (0, 0, 0)$, $B = (1, 2, 3)$, $C = (2, 3, 3)$ and $D = (1, 1, 0)$.

Solution: The two sides adjacent to vertex A in the parallelogram are given by the vectors $B - A = (1, 2, 3)$ and $D - A = (1, 1, 0)$. The area is the magnitude of the cross product:

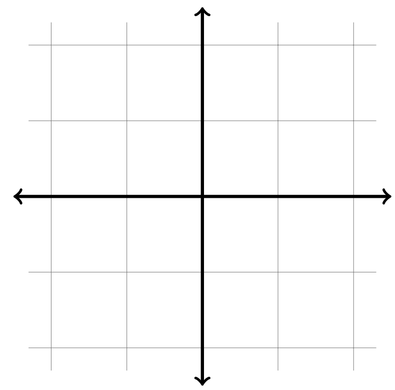
$$||(1, 2, 3) \times (1, 1, 0)|| = || - 3\mathbf{i} + 3\mathbf{j} + \mathbf{k}|| = \sqrt{9 + 9 + 1} = \sqrt{19}.$$

4. (8 points) Draw the regions described by the following polar coordinate inequalities.

(a) $1 \le r \le 2$

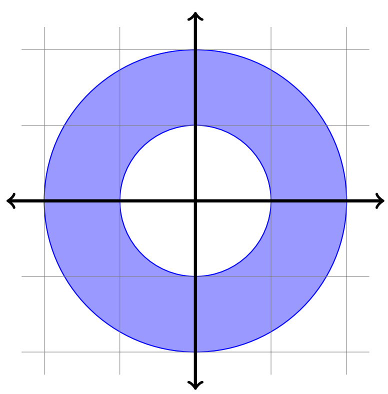


(b) $-1 \le r \le 1$ and $0 \le \theta \le \pi/4$.

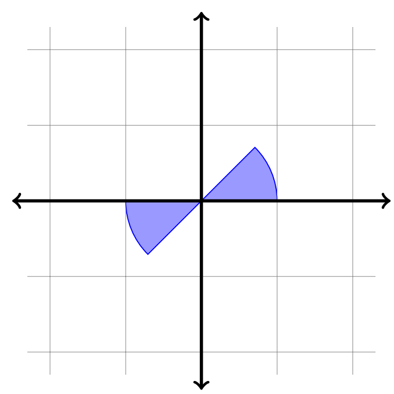


Solution:

(a) $1 \le r \le 2$



(b) $-1 \le r \le 1$ and $0 \le \theta \le \pi/4$.



5. (8 points) Find the slope of the tangent line to the parametric curve $x = \ln t$, $y = 1 + t^2$; at $t = 1$.

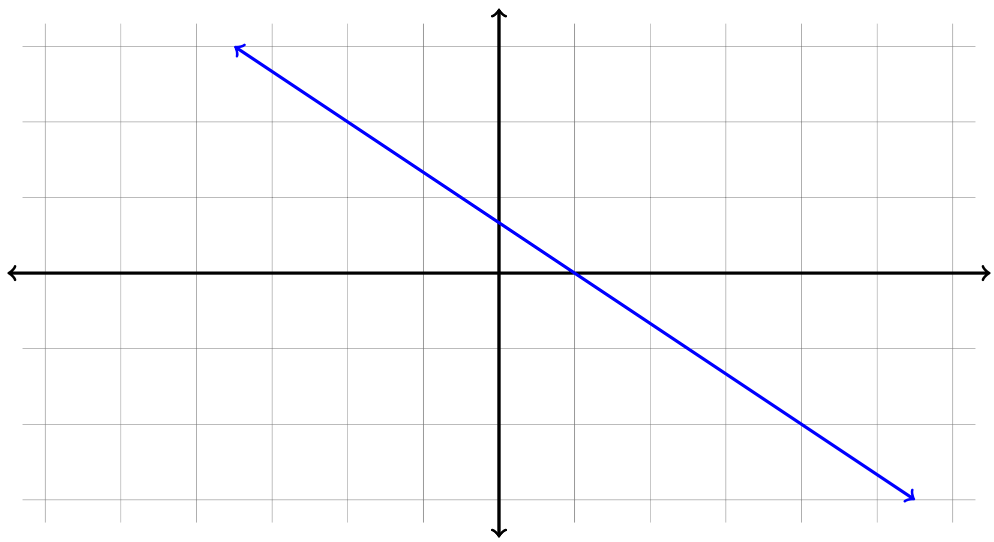
Solution: The slope is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1/t} = 2t^2$$

Therefore, at $t = 1$, $dy/dx = 2$.

6. (8 points) Sketch the graph defined by $\mathbf{v} + t\mathbf{w}$ where $\mathbf{v} = (4, -2)$ and $\mathbf{w} = (-3, 2)$.

Solution:



7. (4 points) Find one normal vector to the plane $x - 2y + 3z = -4$.

Solution: The coefficients of the scalar equation of a plane are the entries of a normal vector:

$$(1, -2, 3).$$

8. (8 points) Does the plane defined by the normal vector $\mathbf{n} = (4, -3, 2)$ that passes through the point $\mathbf{r_0} = (1, 0, 0)$ contain the point $P = (1, 2, 3)$? Why or why not?

Solution: The vector equation for this plane is $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$. Let $\mathbf{r} = P = (1, 2, 3)$ and plug-in to see that

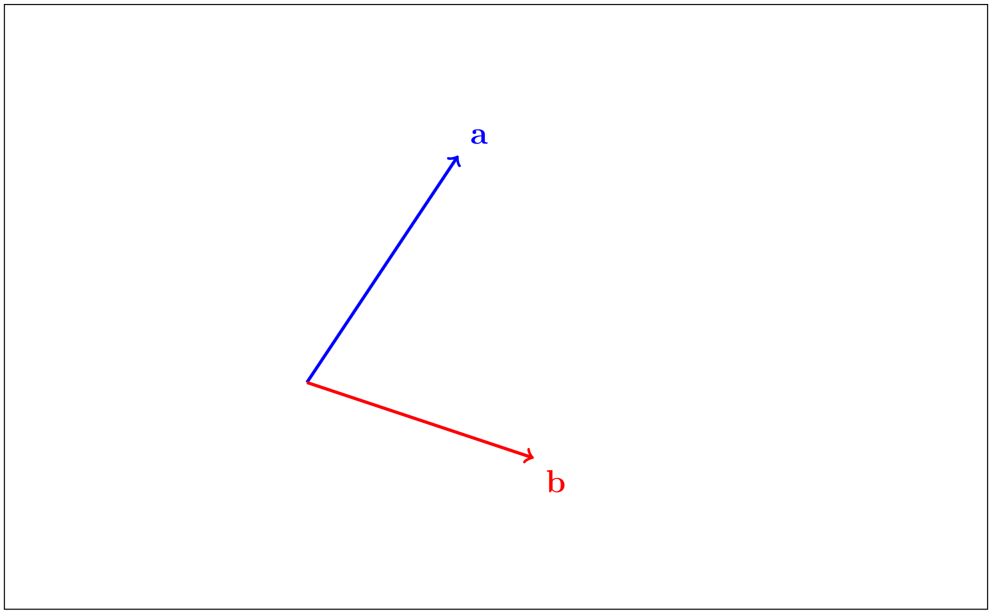
$$(4, -3, 2) \cdot ((1, 2, 3) - (1, 0, 0)) = (4, -3, 2) \cdot (0, 2, 3) = 0 - 6 + 6 = 0,$$

therefore P is in the plane because it satisfies the equation.

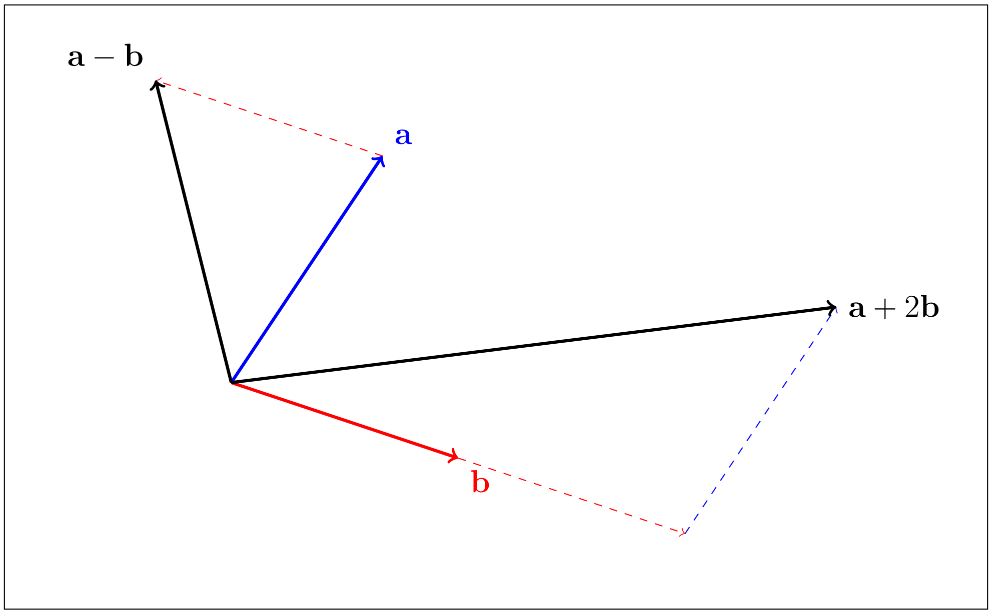
9. (8 points) Using the vectors $\mathbf{a}$ and $\mathbf{b}$ shown below, draw the following vectors.

(a) $\mathbf{a} + 2\mathbf{b}$

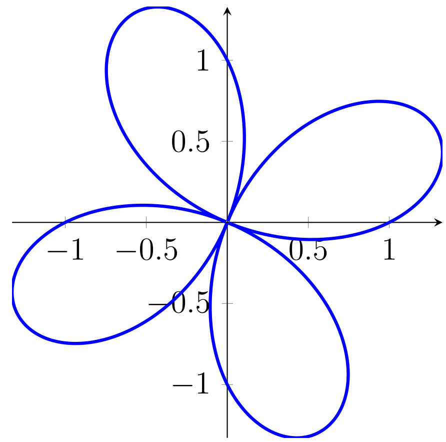
(b) $\mathbf{a} - \mathbf{b}$



Solution:



10. (12 points) Find the area inside the polar curve $r = \sqrt{1 + \sin(4\theta)}$ (graph shown below).

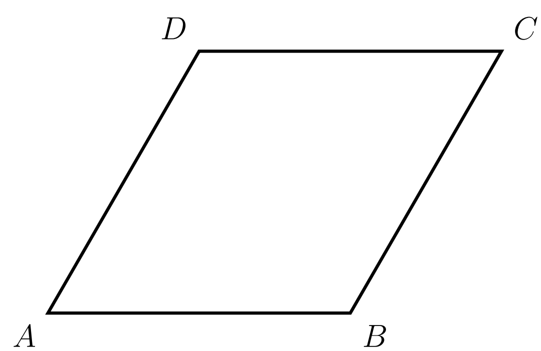


Solution:

$$\text{Area} = \tfrac{1}{2}\int_0^{2\pi} r^2\,d\theta = \tfrac{1}{2}\int_0^{2\pi} 1 + \sin(4\theta)\,d\theta =$$

$$\tfrac{1}{2}\left[\theta - \tfrac{1}{4}\cos(4\theta)\right]_0^{2\pi} = \tfrac{1}{2}\left[(2\pi - \tfrac{1}{4}\cos(8\pi)) - (0 - \tfrac{1}{4}\cos(0))\right] = \pi$$

11. (12 points) Show that the diagonals of a rhombus are always perpendicular. Hint: let $\mathbf{x} = \overrightarrow{AB}$ and $\mathbf{y} = \overrightarrow{AD}$.



Solution: The two diagonals are given by the vectors $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$. To see if they are orthogonal, we compute the dot product:

$$(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = \mathbf{x}\cdot\mathbf{x} - \cancel{\mathbf{x}\cdot\mathbf{y}} + \cancel{\mathbf{y}\cdot\mathbf{x}} - \mathbf{y}\cdot\mathbf{y}$$

$$= \mathbf{x}\cdot\mathbf{x} - \mathbf{y}\cdot\mathbf{y} = ||\mathbf{x}||^2 - ||\mathbf{y}||^2.$$

Since $||\mathbf{x}|| = ||\mathbf{y}||$, the dot product above is zero and therefore the diagonals are orthogonal.

12. (12 points) Let $\mathbf{a} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Compute the values of each expression, or explain why the expression does not make sense.

(a) $||\mathbf{ab}||$

> **Solution:** Does not make sense because you don't multiply vectors.

(b) $||\mathbf{a}||\,||\mathbf{b}||\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

> **Solution:** $||\mathbf{a}||\,||\mathbf{b}||\cos\theta = \mathbf{a}\cdot\mathbf{b}$, therefore we just compute $(-3,4,1)\cdot(1,1,1) = 2$.

(c) $\mathbf{a}^2$

> **Solution:** Doesn't makes sense because you can't square vectors.

(d) $\mathbf{a} \times \mathbf{b}$

> **Solution:**
>
> $$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 4 \\ 1 & 1 \end{vmatrix}\mathbf{k} = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}.$$