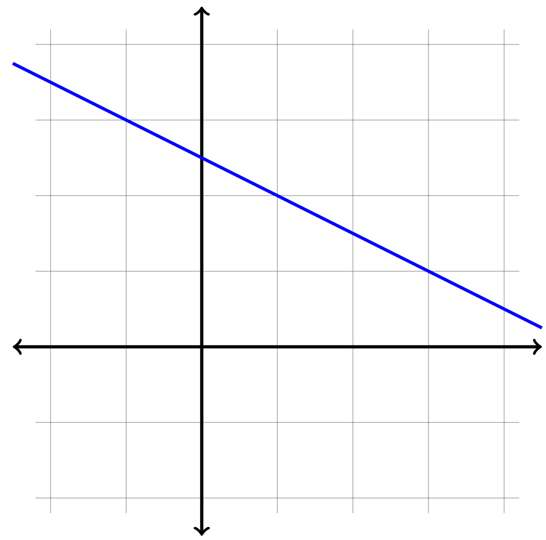
Math 242 **Midterm 2 Solutions**

1. (8 points) Find a vector valued function $\mathbf{r}(t)$ which parametrizes the line shown below.



Solution: Most common mistake was not correctly using the formula $\mathbf{r}_0 + t\mathbf{v}_0$ for a line. People who converted slope-intercept formulas got the right answer too, but were more likely to make a mistake.

Here $\mathbf{r}_0 = (1, 2)$ works, and the direction vector $\mathbf{v}_0 = (2, -1)$ works. So $(1, 2) + (2, -1)t$ is one correct solution.

2. (8 points) Find a formula for the tangent line to the curve with parametric equations $x = t^5$, $y = t^4$, and $z = t^3$ at the point $(1, 1, 1)$.

Solution: Most common mistake was including variables in the direction vector which should be constant for a line.

Correct Solution: $(1, 1, 1) + (5, 4, 3)t$.

3. (8 points) Convert the equation $\rho\cos(\phi) = 2$ from spherical coordinates to rectangular coordinates, and describe (in words) the shape of the surface that corresponds to the equation.

Solution: Some people got $z = 2$, but weren't sure what to do with it. You don't need to do anything with it, it is the solution.

Correct Solution: $z = 2$, which is a plane.

4. (8 points) For what values of t does the line $(t, 2t, 1-t)$ intersect the sphere of radius 1 centered at the origin?

Solution: Since this is about a line and a sphere, you need equations for both the line and the sphere!

Correct solution: $t = 0$ and $t = \frac{1}{3}$.

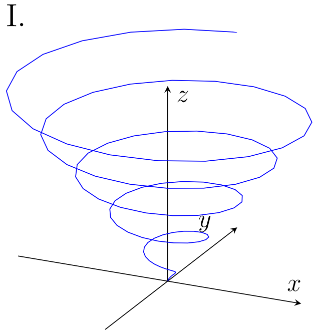
5. (6 points) Match the parametric equations (a) - (c) with the graphs I - III.

   (a) $x = t\cos(t)$, $y = t\sin(t)$, $z = t$

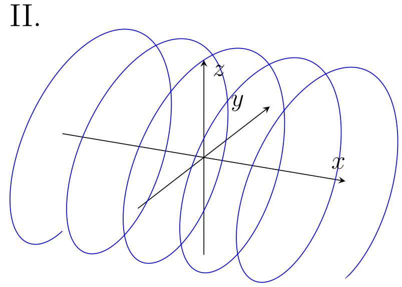
   (b) $x = t$, $y = \cos(5t)$, $z = \sin(5t)$

   (c) $x = \cos(t)$, $y = t$, $z = 2$

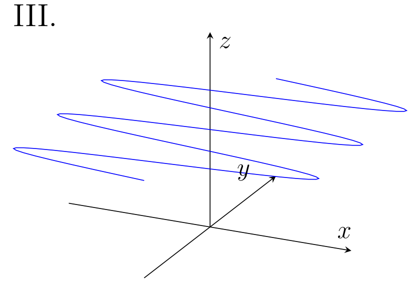
I.



II.



III.



Solution: (a) I., (b) II., (c) III.

6. (12 points) Let $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$.

   (a) Find $\mathbf{r}'(t)$

   (b) Sketch the 2-dimensional curve $\mathbf{r}(t)$.

   (c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ when $t = 0$.

Solution: Almost everyone got (a). For part (b), some people graphed $\mathbf{r}'(t)$ not $\mathbf{r}(t)$. For part (c), some people did not plug-in $t = 0$ to get $\mathbf{r}(0)$ and $\mathbf{r}'(0)$.

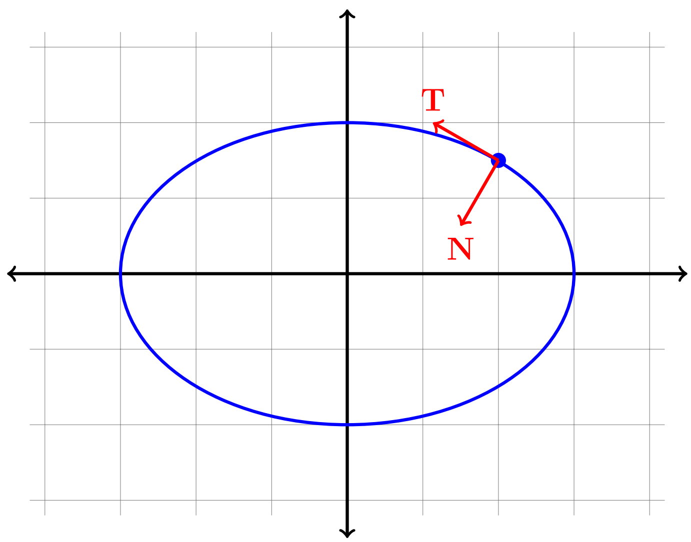
Correct Solution: (a) $\mathbf{r}'(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$, (b) See this Sage graph (c) Draw the vectors $\mathbf{r}(0) = (1, 1)$ and $\mathbf{r}'(0) = (1, -1)$.

7. (8 points) An object has velocity $\mathbf{v}(t) = (1, 3t^2, 2t)$ and initial position $\mathbf{r}(0) = (3, 2, 1)$. Find the acceleration and position as functions of t for the object.

Solution: People forgot the constants when they integrated velocity to get position.

Correct Solution: $\mathbf{a}(t) = (0, 6t, 2)$, $\mathbf{r}(t) = (t + 3, t^3 + 2, t^2 + 1)$.

8. (6 points) A particle is moving counter-clockwise along the path in $\mathbb{R}^2$ shown below. Draw (roughly) the unit normal and unit tangent vectors $\mathbf{N}(t)$ and $\mathbf{T}(t)$ at the point $(2, \frac{3}{2})$.



Solution: The two most common mistakes were to draw the unit normal pointing outwards instead of inwards, and to draw vectors that were too long (unit vectors only have length one).

9. (12 points) Let $f(x, y) = \sin(xy^2)$. Compute the following partial derivatives:

(a) $f_x =$

(b) $f_y =$

(c) $f_{xy} =$

Solution: One large mistake people made was that instead of treating the other variables as constants, they got rid of them entirely. So to find f_x they just got rid of all the y-variables.

Correct Solutions: (a) $y^2 \cos(xy^2)$, (b) $2xy \cos(xy^2)$, (c) $2y \cos(xy^2) - 2xy^3 \sin(xy^2)$.

10. (12 points) Let $g(x, y) = \dfrac{x^2}{y}$.

(a) Find and sketch three different level curves for the surface $z = g(x, y)$ in the same graph. Clearly indicate the level of each curve.

> **Solution:**
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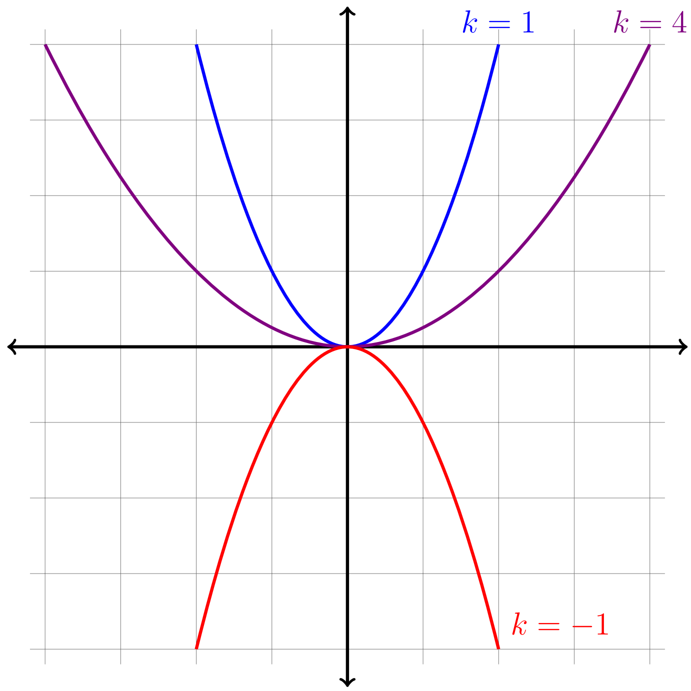
> 
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(b) Explain how you can tell that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{y}$ does not exist.

> **Solution:** Many people said the limit does not exist because it would be $\frac{0}{0}$. That is not why the limit does not exist. Many $\frac{0}{0}$ limits do end up existing. The reason it does not exist is because we get different values depending on which level curve we follow towards the origin.

11. (12 points) Suppose that $\mathbf{T}(t)$ is the unit tangent of a curve $\mathbf{r}(t)$.

(a) Compute $\dfrac{d}{dt}\left(\mathbf{T}(t) \cdot \mathbf{T}(t)\right)$ using the product rule.

> **Solution:** $2\mathbf{T}(t) \cdot \mathbf{T}'(t)$

(b) Explain why $\frac{d}{dt}\left(||\mathbf{T}(t)||^2\right)$ must be zero.

Solution: A common answer was that the derivative of a scalar is always zero. That isn't exactly correct. The derivative of a *constant* is zero. Some scalars are constant and some are not.

Correct Solution: The derivative of a constant is zero, and the magnitude of $\mathbf{T}(t)$ is always the constant 1.

(c) Explain why $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$.

Solution: Many people did not realize that the point of part (c) was simply to combine the answers from part (a) and (b). Several people said that the derivative of a vector valued function is always orthogonal to the original function, but that is not true in general.

Correct Solution: Combining parts (a) and (b) we see that $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$, so the two vectors are orthogonal.