Homework 1 **Quantum Computing**

Let $M_{m,n}$ denote the set of all m-by-n matrices with complex entries. We will abbreviate the notation to M_n when $m = n$. We denote the transpose of $A \in M_{m,n}$ by A^T, the complex conjugate of A by $\bar{A}$, and the conjugate-transpose by A^* (equivalently, $A^\dagger$). We denote the set of n-entry column vectors by $\mathbb{C}^n$. The spectrum of $A \in M_n$, denoted $\sigma(A)$, is the set of all eigenvalues of A.

We will use bra-ket notation for inner and outer-products of two vectors. For two vectors $x, y \in \mathbb{C}^n$, the inner-product is $\langle x|y\rangle = x^*y$ and the outer-product is $xy^* = |x\rangle\langle y|$.

Definition. Let $A \in M_n(\mathbb{C})$. If

1. $A^* = A$, then A is *Hermitian*,
2. $A^* = A^{-1}$, then A is *unitary*,
3. $AA^* = A^*A$, then A is *normal*.

Theorem (The Spectral Theorem)**.** *Every normal matrix $A \in M_n(\mathbb{C})$ is unitarily similar to a diagonal matrix D, that is, there exists a unitary matrix U such that $A = UDU^*$. The columns of U are the eigenvectors of A and the corresponding diagonal entries of D are the eigenvalues of A.*

Exercises

1. Verify that $\langle x|x\rangle = ||x||^2$ for all $x \in \mathbb{C}^n$.
2. Prove that $\langle x|Ay\rangle = \langle A^*x|y\rangle$ for all $x, y \in \mathbb{C}^n$ and $A \in M_n(\mathbb{C})$.
3. Prove that if $A \in M_n$ is Hermitian and λ is an eigenvalue of A, then $\lambda \in \mathbb{R}$.
4. Prove that unitary and Hermitian matrices are normal.
5. Prove that a normal matrix A is unitary if and only if $|\lambda| = 1$ for all $\lambda \in \sigma(A)$.
6. Prove that a normal matrix A is Hermitian if and only if $\lambda \in \mathbb{R}$ for all $\lambda \in \sigma(A)$.
7. Prove that $\operatorname{tr}(Axx^*) = x^*Ax$. In other words, prove that $\langle x|A|x\rangle = \operatorname{tr}(A|x\rangle\langle x|)$.
8. A matrix $A \in M_n$ is positive semidefinite if $\langle x|A|x\rangle \geq 0$ for all $x \in \mathbb{C}^n$. Prove that for any matrix $A \in M_{m,n}$, the matrices A^*A and AA^* are both positive semidefinite.
9. The numerical range of a matrix $A \in M_n$ is the set $W(A) = \{\langle x|A|x\rangle : x \in \mathbb{C}^n, ||x|| = 1\}$. Prove that the numerical range contains the eigenvalues of A.
10. Suppose that $A \in M_n$ is normal, and $x_1, \ldots, x_n$ is an orthonormal basis of eigenvectors of A with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that $A = \lambda_1 x_1 x_1^* + \ldots + \lambda_n x_n x_n^*$.