Homework 2 **Quantum Computing**

For a vector space V, a norm is a function $||\cdot||$ from V to $\mathbb{R}$ with the following properties.

1. **Absolute homogeneity** $||\alpha x|| = |\alpha|||x||$ for all $x \in V$ and $\alpha \in \mathbb{C}$.
2. **Triangle inequality** $||x+y|| \leq ||x|| + ||y||$ for all $x, y \in V$.
3. **Positivity** $||x|| \geq 0$ for all $x \in V$ and $||x|| = 0$ if and only if $x = 0$.

An *inner product space* is a vector space V with an inner product $\langle \cdot | \cdot \rangle$ which satisfies the following properties.

1. **Conjugate symmetry** $\langle x|y\rangle = \overline{\langle y|x\rangle}$ for all $x, y \in V$.
2. **Linearity in second argument** $\langle x|\alpha y + \beta z\rangle = \alpha \langle x|y\rangle + \beta \langle x|z\rangle$ for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$.
3. **Positive definiteness** $\langle x|x\rangle \geq 0$ and $\langle x|x\rangle = 0$ iff $x = 0$.

Exercises

1. Let $p(z) = \alpha_0 + \alpha_1 z + \ldots + \alpha_k z^k$. Let A be an n-by-n normal matrix such that $A = UDU^*$ where D is diagonal and U is unitary. Prove that $p(A) = Up(D)U^*$.
2. Suppose that A is a normal matrix and p is a polynomial. What can you say about the spectrum $\sigma(p(A))$ of $p(A)$?
3. In general, if f is any function that is defined on the spectrum of a normal matrix $A = UDU^*$, then $f(A) = Uf(D)U^*$. Let H be a Hermitian matrix. Prove that e^{iH} is unitary.
4. The Frobenius norm of a matrix is $||A||_2 = \sqrt{\mathrm{tr}\,(A^*A)}$. Prove that the Frobenius norm has the three properties of a norm listed above.
5. The spectral norm of a matrix $|||A|||_2$ is the maximum of $\{||Ax|| : x \in \mathbb{C}^n, x^*x = 1\}$. Prove that the spectral norm satisfies the properties of a norm.
6. For matrices $A, B \in M_{m,n}$, prove that the map $\langle A|B\rangle = \mathrm{tr}\,(A^*B)$ is an inner-product.
7. Prove that the standard inner product on $\mathbb{C}^n$ is unitarily invariant, i.e., for all $x, y \in \mathbb{C}^n$ and all n-by-n unitary matrices, $\langle Ux|Uy\rangle = \langle x|y\rangle$.
8. Prove that the matrix inner-product $\langle A|B\rangle = \mathrm{tr}\,(A^*B)$ is unitarily invariant, i.e., if U and V are unitary matrices, then $\langle UAV|UBV\rangle = \langle A|B\rangle$.
9. Prove that both the Frobenius norm and the spectral norm are unitarily invariant, that is if U and V are unitary matrices then $||A||_2 = ||UAV||_2$ and $|||A|||_2 = |||UAV|||_2$ for all $A \in M_{m,n}$.