Homework 4 **Quantum Computing**

Singular Value Decomposition Every m-by-n matrix A can be expressed

$$A = U\Sigma V^*$$

where U is an m-by-m unitary matrix, V is n-by-n unitary, and Σ is an m-by-n matrix with zero entries except for nonnegative entries $\sigma_1, \ldots, \sigma_k$ ($k = \min\{m, n\}$) on the main diagonal which are called the *singular values* of A.

Exercises

1. Show that A has singular value decomposition $U\Sigma V^*$, then the columns of U are eigenvectors of the matrix AA^*.

2. Show that the columns of V are eigenvectors of A^*A.

3. Show that the singular values of A are the square-roots of the eigenvalues of both A^*A and AA^*.

4. Show that if $A \in M_n$ is normal, then the singular values of A are the absolute values of the eigenvalues of A.

5. Prove that if A is a matrix with real entries, then the singular value decomposition can be chosen so that U, Σ, and V are all real matrices.

6. Let $A \in M_n$. Then $H = \frac{A+A^*}{2}$ and $K = \frac{A-A^*}{2i}$ are known as the real and imaginary parts of A, respectively. Prove that both H and K are Hermitian, and that $A = H + iK$.

7. Recall that $A \in M_n$ is positive semi-definite if and only if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$. Prove that a matrix is positive definite if and only if it is Hermitian and all eigenvalues are nonnegative. (Hint: To show that all positive definite matrices are Hermitian, express A as $H + iK$ in the formula x^*Ax).