## Homework 5

Let  $A \in M_n(\mathbb{C})$  and  $b \in \mathbb{C}^n$ . An **affine linear transformation** is a transformation of the form  $x \mapsto Ax + b$ .

## Exercises

- 1. Prove that the composition of two affine linear transformations is affine linear.
- 2. Prove that if U is a real 2-by-2 unitary matrix, then if det U = 1, then U is a rotation, and if det U = -1, then it is a reflection.
- 3. Suppose that U is a real 3-by-3 unitary matrix and det U = 1.
  - (a) Prove that U must have eigenvalues 1,  $e^{i\theta}$  and  $e^{-i\theta}$  for some  $\theta \in [0, 2\pi)$ .
  - (b) Prove that U is a rotation of  $\mathbb{R}^3$ . Hint: typically U has two complex eigenvectors  $x_1$  and  $x_2$ . Suppose that y is a real vector spanned by  $x_1$  and  $x_2$ . What is Uy?
  - (c) What is its axis of rotation of U? What is the rotation angle?
- 4. Prove that  $\alpha$  is a rational multiple of  $2\pi$  if and only if there is an m such that  $R(\alpha)^m = I$  where  $R(\alpha)$  is the 2-by-2 rotation matrix corresponding to  $\alpha$ .
- 5. Let  $x, y \in \mathbb{C}^2$  be unit vectors, and suppose that the angle between x and y is  $\theta$ . Prove that the corresponding points on the Bloch sphere are separated by an angle of  $2\theta$ .