Complex Analysis Homework #6
Due Wednesday, March 23

Complete any ***two*** *of the following three proofs. You may solve all three for extra-credit.*

1. Suppose that $f(z)$ is an entire function, and $f^{(n)}(z) = 0$ (that is, the n^{th} derivative of $f(z)$) for some positive integer n. Prove that $f(z)$ is a polynomial.

2. Use Cauchy's formula, $f(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(z)}{z_0 - z}\,dz$ to prove the first of **Cauchy's estimates**,
$$|f(z_0)| \leq \max_{|z-z_0|=r} |f(z)|$$
(Hint: See the worksheet I gave you.)

3. Suppose that f is an analytic function and A, m, and R_0 are positive constants such that $|f(z)| \leq A|z|^m$ for all $z \in \mathbb{C}$ with $|z| \geq R_0$. Prove that f is a polynomial of degree at most m. (Hint: Use the Cauchy estimates formula below with $n > m$ and let $r \to \infty$.)

The Cauchy Estimates If $f(z)$ is entire, then
$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|, \qquad n = 0, 1, 2, ...$$
where r is any positive real number.