Review Problems

1. In this problem, U will be the region in the complex plane defined by the inequalities $|z| > 1$ and $-\frac{\pi}{2} < \operatorname{Arg} z < \frac{3\pi}{4}$.

   (a) Draw a sketch of the region U. Is U a connected open set? Is U a simply connected set?

   (b) Suppose that $F(z) = z^2$. If $z = re^{i\theta}$, write a formula for $F(z)$ in complex exponential form. Then sketch the region $F(U)$. Is $F(U)$ simply connected?

   (c) Suppose that $G(z) = 1/z$. Write $G(z)$ in complex exponential form. Sketch $G(U)$. Is $G(U)$ simply connected?

2. Find $\operatorname{Arg} z$ if $z = (1+i)^i$.

3. Describe all solutions of $z^3 = 2i$ algebraic in either rectangular or polar form. The plot these solutions on a graph.

4. Show that $u(x, y) = xy + y$ is a harmonic function. Then find a function $v(x, y)$ such that $u + iv$ is an analytic function. (Note that the function v is called the harmonic conjugate of u.)

5. Use the Residue theorem to calculate $\displaystyle\int_0^\infty \frac{2z^2}{z^4+4}\,dz$.

6. Suppose that $f(z) = \dfrac{1}{z} - \dfrac{1}{\sin z}$. Identify as precisely as possible the type of singularity at $z = 0$. What is the residue at $z = 0$? Find the first two terms of the Laurent series for $f(z)$ centered at $z = 0$.

7. Use geometric series to find a simple expression $S(x)$ for the sum $\displaystyle\sum_{n=1}^{\infty} \frac{e^{inx}}{2^n}$ where x is a real number. Explain why the series converges. What is the imaginary part of $S(x)$? Verify that

$$\frac{2\sin x}{5 - 4\cos x} = \sum_{n=1}^{\infty} \frac{\sin(nx)}{2^n}.$$

8. Find the radius of convergence of the Taylor series expansion centered at $z = i$ of the function $W(z) = \dfrac{e^z - 1}{(z^2 - 1)}$. Give an exact answer. Justify why the series must converge with at least that radius and why it can't have a larger radius. (Actual computation of the series is not practical and is not requested.)

9. Suppose $F(z) = z^3 \left(\dfrac{1}{z+1} + \dfrac{1}{(z-1)^3} \right)$, and C is a simple closed curve which does not pass through 1 or -1. What are all possible values of $\oint_C F(z)\, dz$ (and why)? Sketch examples of curves C which will give each value you list.

10. Let $G(z) = z^5 + 5z^2 + e^z$. How many zeroes (counting multiplicity) does G have in the annular region $1 < |z| < 2$?

11. Suppose that $F(z)$ is an analytic function defined in an open disc centered at 0 and

$$|F^{(k)}(0)| \leq 7$$

for every positive integer k. Explain why $F(z)$ must be equal to an entire function. **Hint:** what can you say about the power series for $F(z)$?