Homework #2 Partial Solutions — Topology

Section 1.3

7. Consider the reciprocal function $r : (0, 1) \to \mathbb{R}$ defined by $r(x) = \frac{1}{x}$.

   (a) The greatest stretching occurs close to $x = 0$.

   (b) Assume that $y > x$ so that $|r(x) - r(y)| = r(x) - r(y)$. Note that
   $$r(x) - r(y) = \frac{1}{x} - \frac{1}{y} = \frac{y - x}{xy}.$$
   To get a lot of stretching, we want the ratio $|r(x) - r(y)|/|x - y|$ to be very large. Note that
   $$\frac{|r(x) - r(y)|}{|x - y|} = \frac{1}{xy}$$
   so if x and y are both small, the stretching can be as large as you want.

   (c) Claim: $r(x)$ is continuous. Fix any $\epsilon > 0$, and let $\delta = x^2\epsilon/2$. Then if $|x - y| \leq \delta$, then by the ratio in part (b), we see that
   $$|r(x) - r(y)| \leq \frac{\delta}{xy} \leq \frac{\epsilon x^2}{2xy} = \frac{\epsilon x}{2y}$$
   and as long as y is sufficiently close to x, $2y > x$ so $|r(x) - r(y)| \leq \epsilon$. This proves that $r(x)$ is continuous on $(0, \infty)$.

9. Consider a function $f : X \to Y$ such that $d(f(x_1), f(x_2)) \leq d(x_1, x_2)$ for all $x_1, x_2 \in X$. Any such function is called **nonexpansive**. Prove that all nonexpansive maps are continuous.

   Suppose that $\epsilon > 0$. Let $\delta = \epsilon$. If $d(x_1, x_2) < \delta$, then $d(f(x_1), f(x_2)) < \epsilon = \delta$. Therefore f is continuous.

11. Consider the **discrete metric** on any set X defined by
   $$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

1. Claim: If X is a space with the discrete metric, then any function $f : X \to Y$ is continuous.

   No matter what $\epsilon > 0$ is , if $\delta = 1$, then $d(x, y) < \delta$ implies that $x = y$ so $f(x) = f(y)$ and $d(f(x), f(y)) < \epsilon$.

2. Claim: If Y is a space with the discrete metric, then if $f : \mathbb{R} \to Y$ is continuous, f must be a constant function.

   If f is continuous, then for every $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$. Suppose that $\epsilon = 1$. Then there is a $\delta > 0$ such that if $|x - y| < \delta$, then $d(f(x), f(y)) < 1$ so $f(x) = f(y)$. Suppose that $x, y \in \mathbb{R}$ are farther than δ apart. Then there is a sequence $x = x_1, x_2, \ldots x_n = y$ such that $d(x_i, x_{i+1}) < \delta$ for all i, and therefore $f(x_i) = f(x) = f(y)$ for all i. Thus f must be a constant function.